Answers to Exercises with Matrices

1. $\begin{bmatrix} 11 & 22 \\ 33 & 44 \\ 55 & 66 \end{bmatrix}$
2. $1 \cdot 3 + 2 \cdot 4 = 11$
3. $\begin{bmatrix} 11 & 2 \end{bmatrix}$
4. $\begin{bmatrix} 11 & 2 \\ 110 & 20 \end{bmatrix}$
5. no answer – can't multiply a 1x3 matrix by a 2x1 matrix
6. $\begin{bmatrix} 3 & 6 & 21 \\ 4 & 8 & 28 \end{bmatrix}$
7. $\begin{bmatrix} 18 & 21 & 24 \\ 5 & 10 & 15 \end{bmatrix}$
8. $\begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$
9. $\begin{bmatrix} 5 & 10 \\ 15 & 20 \\ 25 & 30 \end{bmatrix}$
10. $\begin{bmatrix} -1 & -2 \\ -3 & -4 \\ 5 & 6 \end{bmatrix}$
11. $\begin{bmatrix} -9 & -18 \\ -27 & -36 \\ -45 & -54 \end{bmatrix}$
12. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$
13. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ (the superscript T means transpose, same as in question 12)
14. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$
15. x=0 and y=1
16. x=5 and y=2
17. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, since the identity matrix times itself equals the identity matrix.

18. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}^{-1}$, so $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by definition of an inverse.

    Hence $\begin{bmatrix} 1a-3b & 2a+4b \\ c-3d & 2c+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Solve 1a-3b=1 and 2a+4b=0 to get

    $a=\frac{2}{5}$, $b=\frac{-1}{5}$. Solve c-3d=0 and 2c+4d=1 to get $c=\frac{3}{10}$, $d=\frac{1}{10}$. Hence the answer is

    $\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$. Verify this! $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so indeed

    $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$.

19. You can check the rules for determinants in the text:
    a. $1 \cdot 4 - 3 \cdot 2 = -2$, not singular
    b. 0, singular
    c. –30, not singular
    d. –30, not singular
20. square matrix
21. column matrix or column vector (or just "vector")
22. row matrix or row vector
23. identity matrix
24. diagonal matrix
25. symmetric matrix
26. permutation matrix
27. upper triangular matrix
28. lower triangular matrix
29. $\begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$
30. $\begin{bmatrix} \frac{1}{n}\sum x_i \\ \frac{1}{n}\sum x_i \\ \vdots \\ \frac{1}{n}\sum x_i \end{bmatrix} = \begin{bmatrix} \overline{x} \\ \overline{x} \\ \vdots \\ \overline{x} \end{bmatrix}$ where $\overline{x}$ is the mean of x_1 through x_n
31. $\begin{bmatrix} x_1 - \overline{x} \\ x_2 - \overline{x} \\ \vdots \\ x_n - \overline{x} \end{bmatrix}$

32. $\left[\mathbf{I}-\frac{1}{n}\mathbf{ii}'\right]\mathbf{x}=\mathbf{Ix}-\frac{1}{n}\mathbf{ii}'\mathbf{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}-\begin{bmatrix}\overline{x}\\\overline{x}\\\vdots\\\overline{x}\end{bmatrix}=\begin{bmatrix}x_1-\overline{x}\\x_2-\overline{x}\\\vdots\\x_n-\overline{x}\end{bmatrix}$

33. $\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

34. $\mathbf{M}\equiv\left[\mathbf{I}-\frac{1}{n}\mathbf{ii}'\right]=\begin{bmatrix}1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n}\\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n}\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n}\end{bmatrix}$

35. $\sum x_i$

36. $\mathbf{i}'(\mathbf{Mx})=\sum_{i=1}^{n}x_i-n\overline{x}=n\overline{x}-n\overline{x}=0$ (for **Mx**, plug in your four-previous answer)

37. **M**

38. $\sum x_i^2$

39. $\sum\left(x_i-\overline{x}\right)^2$

40. $\sum x_i y_i$

41. $\sum\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$

42. $\frac{1}{n-1}\sum\left(x_i-\overline{x}\right)\left(y_i-\overline{y}\right)$

43. $\begin{bmatrix}x_{11}-\overline{x}_1 & x_{12}-\overline{x}_2 & \cdots & x_{1k}-\overline{x}_k\\ x_{21}-\overline{x}_1 & x_{22}-\overline{x}_2 & \cdots & x_{2k}-\overline{x}_k\\ \vdots & \vdots & \ddots & \vdots\\ x_{n1}-\overline{x}_1 & x_{n2}-\overline{x}_2 & \cdots & x_{nk}-\overline{x}_k\end{bmatrix}$ where $\overline{x}_i$ is the mean of x_{1i} through x_{ni}

44. $\begin{bmatrix}\sum\left(x_{i1}-\overline{x}_1\right)^2 & \sum\left(x_{i1}-\overline{x}_1\right)\left(x_{i2}-\overline{x}_2\right) & \cdots & \sum\left(x_{i1}-\overline{x}_1\right)\left(x_{ik}-\overline{x}_k\right)\\ \sum\left(x_{i2}-\overline{x}_2\right)\left(x_{i1}-\overline{x}_1\right) & \sum\left(x_{i2}-\overline{x}_2\right)^2 & \cdots & \sum\left(x_{i2}-\overline{x}_2\right)\left(x_{ik}-\overline{x}_k\right)\\ \vdots & \vdots & \ddots & \vdots\\ \sum\left(x_{ik}-\overline{x}_k\right)\left(x_{i1}-\overline{x}_1\right) & \sum\left(x_{ik}-\overline{x}_k\right)\left(x_{i2}-\overline{x}_2\right) & \cdots & \sum\left(x_{ik}-\overline{x}_k\right)^2\end{bmatrix}$

45. $\begin{bmatrix}\mathrm{var}(x_{\cdot 1}) & \mathrm{cov}(x_{\cdot 1},x_{\cdot 2}) & \cdots & \mathrm{cov}(x_{\cdot 1},x_{\cdot k})\\ \mathrm{cov}(x_{\cdot 2},x_{\cdot 1}) & \mathrm{var}(x_{\cdot 2}) & \cdots & \mathrm{cov}(x_{\cdot 2},x_{\cdot k})\\ \vdots & \vdots & \ddots & \vdots\\ \mathrm{cov}(x_{\cdot k},x_{\cdot 1}) & \mathrm{cov}(x_{\cdot k},x_{\cdot 2}) & \cdots & \mathrm{var}(x_{\cdot k})\end{bmatrix}$ where

$\mathrm{cov}(x_{\cdot a},x_{\cdot b})=\frac{1}{n-1}\sum\left(x_{ia}-\overline{x}_a\right)\left(x_{ib}-\overline{x}_b\right)$ and $\mathrm{var}(x_{\cdot a})=\mathrm{cov}(x_{\cdot a},x_{\cdot a})$

46. **I**

47. $\mathbf{I}$
48. $\mathbf{C}$
49. $\mathbf{I}$
50. $\mathbf{D}'\mathbf{C}'$
51. $\mathbf{D}^{-1}\mathbf{C}^{-1}$
52. $\mathbf{A}^{-1}\mathbf{A}' = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, since $\mathbf{A}$ is symmetric
53. $\mathbf{C}^{-1}\mathbf{C}'$ (this cannot be simplified, since $\mathbf{C}$ is not necessarily symmetric)
54. $\mathbf{B}^{-1}(\mathbf{AB})'\mathbf{A}^{-1} = \mathbf{B}^{-1}\mathbf{B}'\mathbf{A}'\mathbf{A}^{-1} = \mathbf{B}^{-1}\mathbf{BAA}^{-1} = \mathbf{II} = \mathbf{I}$
55. $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1.5 \\ 1.5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
56. $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
57. not a quadratic form – the term $7x_1$ is not quadratic
58. $[x_1][1][x_1]$
59. $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
60. not a quadratic form
61. not a quadratic form
62. $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & 0.5 & -5/2\pi \\ 0.5 & 1 & -6 \\ -5/2\pi & -6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
63. $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \alpha & \delta/2\rho & \phi/2 \\ \delta/2\rho & \beta & \sigma/2 \\ \phi/2 & \sigma/2 & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
64. The answer to the first step is $a_1u_1 + a_2u_2 + \cdots + a_nu_n$. The answer to the second step is $\dfrac{\partial \mathbf{a}'\mathbf{u}}{\partial u_i} = \dfrac{\partial(a_1u_1 + a_2u_2 + \cdots + a_nu_n)}{\partial u_i} = a_i$. The answer to the third step is therefore $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}'$. By definition, $\dfrac{\partial f(\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial f(\mathbf{u})}{\partial u_1} & \cdots & \dfrac{\partial f(\mathbf{u})}{\partial u_n} \end{bmatrix}'$, so applying this formula to the function $f(\mathbf{u}) = \mathbf{a}'\mathbf{u}$, you see that you have proved the formula $\dfrac{\partial(\mathbf{a}'\mathbf{u})}{\partial \mathbf{u}} = \mathbf{a}$.
65. First compute the scalar $\mathbf{u}'\mathbf{Au}$, and compute its vector of derivatives. Then compute $2\mathbf{Au}$ and see if you get the same answer.

$$\begin{aligned} \mathbf{u}'\mathbf{Au} &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} a_{11}u_1 + a_{21}u_2 & a_{12}u_1 + a_{22}u_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= a_{11}u_1^2 + a_{21}u_1u_2 + a_{12}u_1u_2 + a_{22}u_2^2 \end{aligned}$$

$$\frac{\partial(\mathbf{u}'\mathbf{A}\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial(\mathbf{u}'\mathbf{A}\mathbf{u})}{\partial u_1} \\ \dfrac{\partial(\mathbf{u}'\mathbf{A}\mathbf{u})}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 2a_{11}u_1 + (a_{21}+a_{12})u_2 \\ (a_{21}+a_{12})u_1 + 2a_{22}u_2 \end{bmatrix} \quad (1)$$

$$2\mathbf{A}\mathbf{u} = 2\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2a_{11}u_1 + 2a_{12}u_2 \\ 2a_{21}u_1 + 2a_{22}u_2 \end{bmatrix} \quad (2)$$

Comparing (1) and (2), it is apparent that they are equal if $a_{21} + a_{12} = 2a_{12}$ and $a_{21} + a_{12} = 2a_{21}$. This can be true only if $a_{12} = a_{21}$. Thus (1) and (2) are equal if and only if **A** is symmetric. Thus $\frac{\partial(\mathbf{u}'\mathbf{A}\mathbf{u})}{\partial \mathbf{u}} = 2\mathbf{A}\mathbf{u}$ if and only if **A** is symmetric.

66. $$\frac{\partial \mathbf{f}}{\partial \mathbf{u}'} = \begin{bmatrix} \dfrac{\partial f_1(\mathbf{u})}{\partial u_1} & \dfrac{\partial f_1(\mathbf{u})}{\partial u_2} & \dfrac{\partial f_1(\mathbf{u})}{\partial u_3} \\ \dfrac{\partial f_2(\mathbf{u})}{\partial u_1} & \dfrac{\partial f_2(\mathbf{u})}{\partial u_2} & \dfrac{\partial f_2(\mathbf{u})}{\partial u_3} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 2u_1u_2^3u_3^4 & 3u_1^2u_2^2u_3^4 & 4u_1^2u_2^3u_3^3 \end{bmatrix}$$

67. $$\frac{\partial^2 f(\mathbf{u})}{\partial \mathbf{u}\partial \mathbf{u}'} = \begin{bmatrix} \dfrac{\partial^2 f(u)}{\partial u_1^2} & \dfrac{\partial^2 f(u)}{\partial u_2 \partial u_1} & \dfrac{\partial^2 f(u)}{\partial u_3 \partial u_1} \\ \dfrac{\partial^2 f(u)}{\partial u_1 \partial u_2} & \dfrac{\partial^2 f(u)}{\partial u_2^2} & \dfrac{\partial^2 f(u)}{\partial u_3 \partial u_2} \\ \dfrac{\partial^2 f(u)}{\partial u_1 \partial u_3} & \dfrac{\partial^2 f(u)}{\partial u_2 \partial u_3} & \dfrac{\partial^2 f(u)}{\partial u_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4u_3^3 \\ 0 & 4u_3^3 & 12u_2u_3^2 \end{bmatrix}$$

68. When answering this question, it helps if you remember quadratic forms.

$$\mathbf{v}'\mathbf{A}\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4u_3^3 \\ 0 & 4u_3^3 & 12u_2u_3^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2v_1^2 + 8u_3^3v_2v_3 + 12u_2u_3^2v_3^2$$

69. Plug the value given for **u** into the above answer, yielding $\mathbf{v}'\mathbf{A}\mathbf{v} = 2v_1^2$. This is positive or zero for all vectors of real numbers **v**, $\mathbf{v}'\mathbf{A}\mathbf{v} \geq 0$. (However $\mathbf{v}'\mathbf{A}\mathbf{v} > 0$ is not correct, for example $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \neq \mathbf{0}$ yields $\mathbf{v}'\mathbf{A}\mathbf{v} = 0$.)

70. The first order condition is $\frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial f(\mathbf{u})}{\partial u_1} \\ \dfrac{\partial f(\mathbf{u})}{\partial u_2} \\ \dfrac{\partial f(\mathbf{u})}{\partial u_3} \end{bmatrix} = \begin{bmatrix} 2u_1 \\ u_3^4 \\ 4u_2u_3^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus, $2u_1 = 0$, $u_3^4 = 0$, and $4u_2u_3^3 = 0$. Solving yields $u_1 = 0$, $u_3 = 0$, and $u_2 =$ anything. So for any k, $\tilde{\mathbf{u}} = \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix}$ is a candidate to yield a minimum of the function. Letting **A** denote

the Hessian matrix obtained previously, and plugging in $\mathbf{u}=\tilde{\mathbf{u}}$, yields $\mathbf{v}'\mathbf{A}\mathbf{v} = 2\mathrm{v}_1^2$ as in the previous question. This is ≥0 for all **v**≠**0**. However it is not >0 for all **v**≠**0**, as demonstrated in the previous answer. Hence form B of the second order condition cannot guarantee that $\tilde{\mathbf{u}}$ yields a local minimum of the function. Were $\mathbf{v}'\mathbf{A}\mathbf{v}$ sometimes positive and sometimes negative, then you would know that $\tilde{\mathbf{u}}$ yields neither a maximum nor a minimum of the function, but this is not true either. So in this case, form B of the second order condition cannot tell you whether or not $\tilde{\mathbf{u}} = \begin{bmatrix} 0 \\ \mathrm{k} \\ 0 \end{bmatrix}$ yields a minimum. (You can be sure it is not a maximum.) In fact it is not a minimum; for example $f\left(\begin{bmatrix} 0 & \mathrm{k} & 0 \end{bmatrix}'\right) = 0$ but $f\left(\begin{bmatrix} 0 & -2 & 2 \end{bmatrix}'\right) = -8$.