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SELF CONTAINMENT RADIUS FOR ROTATING PLANAR FLOWS, SINGLE-SIGNED VORTEX GAS AND ELECTRON PLASMA

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A low temperature relation $R^2 = \Omega\beta/4\pi\mu$ between the radius R of a compactly supported 2D vorticity (plasma density) field, the total circulation Ω (total electron charge) and the ratio μ/β (Larmor frequency), is rigorously derived from a variational Principle of Minimum Energy for 2D Euler dynamics. This relation and the predicted structure of the global minimizers or ground states are in agreement with the radii of the most probable vorticity distributions for a vortex gas of N point vortices in the unbounded plane for a very wide range of temperatures, including $\beta = O(1)$. In view of the fact that the planar vortex gas is representative of many 2D and 2.5D statistical mechanics models for geophysical flows, the Principle of Minimum Energy is expected to provide a useful method for predicting the statistical properties of these models in a wide range of low to moderate temperatures.

To the memory of Henri Poincaré,
on the 150th anniversary of his birth

Contents

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1. Introduction

This paper derives an elegant formula (2.1) for 2D Euler flows in the unbounded plane, which holds between inverse temperature β, chemical potential μ, total circulation Ω and the radius R of the compact support of steady-state vorticity when $\beta = k\mu \to \infty$ with fixed $k > 0$. It will be clear from the results in this paper that vortex dynamics on the unbounded plane differs significantly from any bounded domain formulation. The key to self-confinement in the unbounded plane lies in the notion of an angular momentum which is conserved by the dynamics because of $SO(2)$ invariance of the vortex gas. Both in electron plasma [32] and geophysical flows [8] on unbounded domains, the confinement of the vorticity or charge can be expressed in terms of a self-containment radius, which is clearly a very useful quantity for which it will be desirable to have a closed form expression. The approach we take is based on the observation that the minimizers of the internal energy are compactly supported when the angular momentum is present.

At this point the reader could well ask why there is an inverse temperature β in the formulation since we are merely minimizing an augmented energy functional in the framework of the continuum Euler model on the unbounded plane. The reason for this lie in the dual origin of the problem at hand, namely, the above dynamical question of steady-states of the Euler equations in the unbounded plane, and the mean field continuum limit of the Onsager Vortex Gas. The latter is an equilibrium statistical mechanics model that consists of N point vortices of like sign in thermal equilibrium with infinite reservoirs of energy and angular momentum. In other words, Onsager's Vortex Gas is based on a Gibbs ensemble that is canonical in both the flow kinetic energy H_N and the angular momentum or moment of vorticity Γ_N.

For rotating 2D flows, it is arguable that the first point of view is primary. In this first point of view, there is no need for thermodynamics and only the ratio $\sigma = \mu/\beta$ plays a physical role as the rate of rotation. On the other hand, the second point of view is perhaps more natural in the setting of cylindrical electron plasmas where the discrete vortices correspond in cross section to the long lines of charge in the plasma. Since the main aim of this paper is to rigorously derive the self-containment formula from 2D Euler dynamics, we will focus on the first point of view.

The same result could also be obtained from the recent technical results on the mean field limit for the Onsager Vortex Gas discussed next. In view of this paper's secondary aim to provide a detailed summary of the recent work on the Onsager Vortex Gas, we will include such a review of the known results, a heuristic derivation of the radius formula in section 2 and the statement of an important Open Problem on the Onsager Vortex Gas.

The existence and asymptotic exactness of this mean field limit was accomplished in the 1990s by Caglioti et al [3] and Kiessling [18]. Subsequently, the convergence of the finite temperature mean field theory to the zero temperature ground state of the single-sign vortex gas in the unbounded plane was established by Kiessling and Spohn [19]. This zero temperature ground state is the minimizer of the augmented energy functional in the first point of view and is related to Ginibre's exact expressions from random matrix theory.

This paper is organized as follows: In section (1.1)-(1.3) we give an overview of known results for both the bounded and unbounded vortex gas problem. Section 2 discusses some numerical as well as heuristic results. The main theorem is proved in section 3. Section 4 presents the results of Monte Carlo simulations which can be explained by our theorem. The proof that the global minimizers of the energy functional (with angular momentum) must be compactly supported and radially symmetric will be reported in another paper.

1.1. Open Problem for the Vortex Gas

The only open problem left in the case of the single-sign vortex gas is an exact closed form partition function for the finite N vortex gas at finite temperatures since Ginibre's result holds only for a single special value of the temperature. This problem is represented by the search for an exact expression of the partition function first introduced by Onsager [28], [6]

$$Z_N(\beta, \mu) = \int_{R^2} dz_1 ..... \int_{R^2} dz_N \exp(-\beta H_N) \exp(-\mu \Gamma_N) \tag{1.1}$$

where β and μ are the Lagrange multipliers associated with respectively the kinetic energy

$$H_N = -\frac{\lambda^2}{2\pi} \sum_{j<k} \ln |z_j - z_k|$$

where $z_j \in R^2$ is the position of particle j in the plane, and moment of vorticity ("angular momentum")

$$\Gamma_N = \lambda \sum_{j=1}^{N} |z_j|^2.$$

In other words, β is the inverse of a temperature for the vortex gas and μ is a chemical potential. The parameter μ is related to the rate of spin of a background flow in which the point vortices are immersed. The parameter μ/β is related to the Larmor gyrofrequency in the electron plasma [30], [32]. A complete solution of this open problem will clearly supersede many of the technical results obtained recently on the exactness of the mean field limit.

1.2. Main results on the Vortex Gas: 1950–2000

Although the partition function Z_N is unknown for most values of β and μ, Ginibre [14] derived an exact solution for the one-particle reduced distribution function

$$G_1^N(z) = \exp(-|z|^2) \sum_{k=0}^{N-1} \frac{|z|^{2k}}{k!} \to 1 \text{ as } N \to \infty \tag{1.2}$$

when $\lambda_j = 1$, $\beta = 2$ and $\mu = 1$. His method was based on the Wigner-Dyson model of the statistics of the spectra of random matrices [14], [7], [12]. From G_1^N we infer the distribution of a point vortex gas of N identical vortices in the canonical ensemble at an inverse temperature $\beta = 2$ and chemical potential $\mu = 1$ has the form of a nearly uniform density in a disk of radius $R \sim \sqrt{N}$ and exponentially decaying density outside. Ginibre's result for any finite N suggests that in the nonextensive continuum limit of $N \to \infty$, the most probable vorticity distribution has radially symmetric compact support, which is moreover, uniform in a disk of radius R.

To summarize the substantial development in point vortex statistics since the classic 1947 paper of Onsager [28], we outline the main lines of approach and main results. Many workers have contributed to various aspects of this problem; we include here several references on topics which are not directly related to the single-sign vortex gas problem. On the key topic of negative temperatures, we refer to the papers of Joyce and Montgomery [26], Lundgren and Pointin [23] and Eyink and Spohn [10]. A mixed sign vortex gas at negative temperatures attain very high energies by separating the physical domain of occupation of the two species of vortices, with the most probable distribution taking the forms of dipoles, quadrupoles and others in a variety of cylindrical containers. The single-sign vortex gas, on the other hand, will develop singularities at negative temperatures; at positive temperatures the single sign gas attains the largest entropy by separating the vortices but the positive value of the chemical potential μ constrains this tendency towards unlimited spreading of the vortices. For this reason, our Monte-Carlo numerical simulations of the single sign vortex gas are performed only at positive temperatures [1].

The second key development is on the thermodynamic limit, which is a problem complicated by the fact that this limit is not the standard one, but instead, is a nonextensive continuum limit when the number $N \to \infty$. The best way to introduce the notion of a nonextensive continuum limit is to invoke one of the precedents of the point vortex gas, namely the point vortex particle method for numerical simulations of 2D problems (cf. Chorin [5], Hou and Lowengrub [17], Beale and Majda [2], Goodman [15], Hald [16]). In this numerical method carried out on for example, a fixed compact flow domain, the number of vortices is increased while their strengths are concurrently decreased to preserve circulations, enstrophy and other flow constrains. Thus, the proper thermodynamic limit is one where the domain size is fixed rather than increasing in step with the number N of vortices in the gas. This requires a concurrent scaling of vortex strengths and / or inverse temperatures β, known as the mean field limits (cf. Lundgren and Pointin [23], Caglioti et al [3], Marchioro and Pulvirenti [24] and Kiessling [18]).

Returning to the single sign vortex gas on the unbounded plane with the angular momentum constraint, the rigorous and highly technical result on the asymptotic exactness of this mean field continuum limit was proved in Chanillo and Kiessling [4] in the context of a class of differential geometric problems on prescribed Gaussian curvatures (cf. also the symmetry and moving plane techniques for nonlinear elliptic PDEs in for example [13], [27]). Apart from this result, another key rigorous result on the convergence of the finite temperature minimizer of the free energy F for this vortex gas to the corresponding zero temperature minimizer, was obtained by Kiessling and Spohn [19]. An equivalent statement of this result is that the most probable vorticity for any finite temperature (the entropy maximizer), that is, the minimizer of the free energy by Planck's famous theorem, is well approximated by the minimizer of the internal energy, at least when $T \ll 1$.

In the context of these two rigorous convergence results, the above exact solution of Ginibre for special values of temperature and chemical potential, and any finite N, converges simultaneously as $N \to \infty$, to the zero temperature minimizer of the free energy F or the minimizer of the internal energy

$$U = \langle H \rangle$$

where the expectation is in the sense of the nonextensive continuum limit of the finite N expectation $\langle \cdot \rangle_N$ defined in terms of (1.1). This then provides an outline of a rigorous proof that the minimizer of U is uniquely that suggested by Ginibre's result, namely

$$\text{it is a uniform vortex distribution on a disk of radius } R. \tag{1.3}$$

The formula for R will be derived below from other considerations.

In this context, what remains to be determined is that for what range of finite temperatures T and finite N, is the zero temperature result (1.3) accurate? Given that there is no known exact closed form expression of the finite N, finite T partition function Z_N in (1.1), our only recourse to probe this question on finite temperatures and finite size scaling, is Monte-Carlo simulations [1].

For completeness, we briefly review the mean field theory in the special case of a bounded domain.

1.2.1. Mean field for bounded domain

The mean field theory based on the scaling,

$$\beta \to \frac{\tilde{\beta}}{N},\ \mu \to \mu,\ \lambda \to \lambda \text{ as } N \to \infty, \tag{1.4}$$

started from the work of Lundgren and Pointin [23] and Montgomery and Joyce [26] on the derivation of an effective model for the one particle distribution function in point vortex statistics.

Other mean field models have been proposed by Miller [25] and Robert [29]; however the Miller-Robert theory have been criticized by Chorin [6] and others. Turkington [31] and Majda [8] have given useful critiques and applications of mean field theory. The nonexistence of an extensive thermodynamic limit for the point vortex gas where $\beta > 0$, $\mu > 0$ and vortex strength λ are fixed as $N \to \infty$, was discussed in Frohlich and Ruelle [11].

In comparison, an important classical result is Chorin's formulation [5] of the vortex method for 2D Euler dynamics where the continuous vorticity field of fixed total circulation is represented by a finite sum of N delta functions of strength

$$\lambda = \Omega/N \text{ as } N \to \infty, \tag{1.5}$$

where Ω is the fixed total circulation.

Subsequent important work including those of Hald [16], Goodman [15], Beale and Majda [2] and Hou et al [17] have derived results which established the convergence of the vortex method to continuum 2D Euler dynamics. It turns out that (1.5) has to be combined with scalings on β and μ to obtain a thermodynamic limit; we shall call this new scaling the vortex method scaling:

$$\beta \to \tilde{\beta}N, \ \mu \to \tilde{\mu}N, \ \lambda \to \Omega/N \text{ as } N \to \infty. \tag{1.6}$$

In the important papers, Caglioti, Lions, Marchioro and Pulvirenti [3], and independently, Kiessling [18] proved the asymptotic exactness of the mean field theory for a vortex gas in a bounded domain, in two different but equivalent nonextensive scaling limits, namely:

$$(1)\ \beta \to \tilde{\beta}N, \ \lambda \to \Omega/N \text{ as } N \to \infty$$

$$(2)\ \beta \to \frac{\tilde{\beta}}{N}, \ \lambda \to \lambda \text{ as } N \to \infty.$$

The first scaling is the counterpart of (1.6) without μ. There is no Lagrange multiplier μ in both these papers because their work is based on a different formulation of the point vortex gas. A fixed bounded region $\Lambda \subset R^2$ is assumed to be the support of the vorticity instead of the unbounded plane discussed here, and the angular momentum constraint is not included in their partition function. Instead of using Onsager's idea of self-bounding the vortex gas by way of the angular momentum, they impose a flow boundary, $\partial\Lambda$. They retain the logarithmic Green's function of the unbounded plane by invoking free boundary conditions on $\partial\Lambda$.

1.3. Main results

The main goal of this paper is thus to characterize the global minimum

$$E_\Omega^0 = \inf_{q \in L_\Omega(R^2)} E_\Omega[q] = E_\Omega[q_0]$$

of the energy functional $E_\Omega[q]$. The existence and uniqueness of compactly supported radially symmetric global minimizers or ground states of $E_\Omega[q]$, which is assumed in this paper, will be proved in another paper. The main result in this paper is theorem 4 which characterizes the unique global minimizer to be the step function radial vorticity distribution $q_0(z) \in L_\Omega(R^2)$, on the way to deriving the central formula (2.1) for the radius R of the support disk of q_0 in terms of the total circulation Ω and the ratio $k = \tilde{\beta}/\tilde{\mu}$.

Verification of this low temperature model will be done using Monte Carlo simulations of the finite vortex gas under a wide range of conditions. This will be discussed in the last section of the paper. The excellent comparison between the simulated values at low and moderate temperatures of maximal radii for a finite vortex gas and the rigorously derived formula (2.1), confirm numerically that the variational theory in this paper is relevant to unbounded vortex gas statistics (cf. [6]) even at $O(1)$ temperatures. In view of the fact that the planar vortex gas is representative of many 2D and 2.5D statistical mechanics models for geophysical flows with angular momentum constraints [8], the Principle of Minimum Energy is expected to provide a useful method for predicting with rigor, the statistical properties of these models in a wide range of low to moderate temperatures. For example, it will be quite exciting to derive the exact formula for potential vorticity support radius in the Heton model, and see its dependence on the Rossby deformation length scale.

2. Auxillary results

We collect here some results to elucidate the relationship between the finite point vortex gas, the nonextensive mean field theory and the minimizers of the energy functional. The heuristic derivation of (2.3) from empirical mean field results is included here to motivate the radius formula (2.1).

2.1. Monte-Carlo simulations

Some of the important work on this problem are numerical in nature [32], [23], [5]; they show at least two distinct classes of most probable vorticity distributions, depending on the values of the Lagrange multipliers β and μ. Direct numerical simulations of the vortex gas, using a Monte-Carlo algorithm, produced Gaussian like radial mean vorticity profiles at high temperatures and low values of the Lagrange multiplier μ, and step function like profiles at low to moderate temperatures and high to moderate values of μ. It is very important to note that the step function like numerical solutions need not have compact support at all.

Recent careful Monte-Carlo simulations [1] of a vortex gas of N points of strength one at low temperatures, confirm this dependence on only the ratio β/μ and N. Our experiments suggest a precise scaling relation between fixed N, β, μ and the most probable radius R of the furthest vortex from the origin, namely

$$R = \sqrt{\frac{N\beta}{4\pi\mu}}. \tag{2.1}$$

This last empirical result is consistent Ginibre's result [14], but signals that the term depending only on the ratio $k = \beta/\mu$ is already highly accurate for N around 10^3, and β and μ ranging from 1 to 10^6. Moreover, it is also consistent with numerical results from the mean field theory discussed below.

On the other hand, for very small values of β and μ, we found empirical radial vorticity profiles that are close to a Gaussian distribution [1]. Mean field theory predicts a different relation than (2.1) while the non-compactly supported radial vorticity distributions in Williamson [32] and Lundgren and Pointin [23] compare well with those in [1] at very small β and μ.

2.2. Heuristic derivation from mean field theory

A heuristic derivation of the central formula (2.1) from mean field theory is discussed here. The virial equation can be easily obtained from rigorous mean field results in [3]. Williamson [32] and Lundgren-Pointin [23] computed a virial relationship between N, β, μ and $\langle r^2 \rangle$, namely:

$$\mu \langle r^2 \rangle - \frac{\beta N}{8\pi} = 1. \tag{2.2}$$

They also presented numerical evidence for step function like radial vorticity density profiles when $\beta = k\mu \to \infty$ with $k > 0$ fixed. Combining these two results from mean field theory, and after using the mean field definition of the expectation $< \cdot >$ to get

$$\langle r^2 \rangle = N^{-1} \int_{D_R} q(x)|x|^2 dx = \frac{R^2_{mft}}{2},$$

we derive from (2.2) heuristically, the interesting new relationship

$$R^2_{mft} = \frac{N\beta}{4\pi\mu} + \frac{2}{\mu}$$

for the radius R_{mft} of step function radial vorticity density. Here D_R is the disc of radius R. This then implies the heuristic asymptotic result:

$$R^2_{mft} = \frac{N\beta}{4\pi\mu} + \frac{2}{\mu} \to \frac{N\beta}{4\pi\mu} \text{ as } \beta = k\mu \to \infty,\ k > 0 \text{ fixed}, \tag{2.3}$$

that is, the central relationship (2.1) should be recovered in the limit of low temperatures and fixed Larmor frequency from mean field theory.

3. Variational Problems for $E[q]$: Derivation of central formula

The full problem of minimizing $E[q]$ on $L^2(R^2)$ and proving that the minimizer has compact support and radial symmetry will not be discussed here. Our aim here is to derive the scaling relations (2.1) and (3.11).

The method requires formulating a suitable variational problem for the augmented energy functional $E[q]$ in the space L^2 of square-integrable vorticity distributions whose compact support are disks D_R of arbitrary radius $0 < R < \infty$. The main reason for the complete solvability of this nonextensive continuum theory, lies in the fact that there is an orthonormal basis of eigenfunctions of the inverse G of the Laplacian which effectively diagonalizes the energy functional $E[q]$ (cf. also Lim [22] for a previous application of this decomposition).

We will discuss two related variational problems in this paper. The first concerns a variational problem on a fixed disk of radius R and free circulation and the second concerns a constrained variational problem with variable domain consisting of disks D_R of arbitrary radius R but fixed total circulation. It is interesting that the form of the relationship between total circulation Ω, β, μ and the radius R of the support of the vorticity density function depends on the type of constraints. For example, when the radius R is fixed but total circulation is free, we get (3.11) in problem 1, which differs from (2.1) obtained in problem 2 where the total circulation is fixed and the radius of the support is free.

In both problems we will use the following definitions. The moment of vorticity for $q(z)$ with respect to the origin is given by the linear functional

$$\Gamma[q] = \int_{D_R} dz\ r^2 q(z). \tag{3.1}$$

The kinetic energy of the ideal fluid flow corresponding to vorticity $q(z)$ is given by the quadratic Hamiltonian functional

$$\begin{aligned} H[q] &= -\frac{1}{4\pi}\int_{D_R} dz\ q(z)\left[\int_{D_R} dz'\ q(z')\ \ln|z-z'|\right] \equiv \\ &\equiv \frac{1}{2}\langle q, G[q]\rangle \end{aligned} \tag{3.2}$$

modulo an additive constant to fix the zero value. Here $G[q]$ is the integral operator

$$G[q](z) = -\frac{1}{2\pi}\int_{D_R} q(z')\ \ln|z-z'|\ dz'.$$

The kernel $K(z, z') = -\frac{1}{2\pi}\ln|z-z'|$ is the Green's function for $-\Delta$ on R^2.

It is easy to show that for any $q \in L^2(D), G[q]$ is also in $L^2(D)$:

$$\begin{aligned} &\frac{1}{4\pi^2}\int_D dz\ \left(\int_D dz'\ q(x')\log|z-z'|^{-1}\right)^2 \leqslant \\ &\leqslant \frac{1}{4\pi^2}\int_D dz\left(\int_D dz'\left(\log|z-z'|^{-1}\right)^2\right)\left(\int_D dz'\ q^2(z')\right) = \\ &= \frac{1}{4\pi^2}\int_D dz\int_D dz'\left(\log|z-z'|^{-1}\right)^2 ||q||^2_{L_2(D)} < \infty. \end{aligned} \tag{3.3}$$

This calculation also show that the norm [21] of the operator G is given by

$$||G|| = \frac{1}{2\pi}\left(\int_D dz \int_D dz' \left(\log|z - z'|^{-1}\right)^2\right)^{1/2} < \infty. \tag{3.4}$$

By the symmetry of the Green's function in the arguments z and z', it follows that G is a compact self-adjoint operator on $L^2(D)$ where $D \subset R^2$ is a compact set. Then by the Spectral Theorem for compact self-adjoint operators [21], there is an orthonormal basis

$$B = \{I_D(x)\} \cup \{\psi_j\}_j \tag{3.5}$$

for $L^2(D)$, consisting of the indicator I_D and eigenfunctions

$$\psi_j = \lambda_j^{-1} G[\psi_j]$$

of G which satisfy the zero total circulation condition,

$$\int_D dz \psi_j(z) = 0. \tag{3.6}$$

This will be used to prove Lemma 2 and 2a on the strict convexity of the energy functional.

After integrating by parts and using the relationship

$$-\Delta\psi = q,$$

between a stream function ψ and the vorticity $q(z)$, the Hamiltonian (3.2) becomes the functional

$$H(\psi) = \frac{1}{2}\int_{D_R} dz\, |\nabla\psi|^2 = \frac{1}{2}\int_{D_R} u^2 dz = \tag{3.7}$$

$$= \frac{1}{2}\int_{D_R} dz q(z) G[q](z) = \frac{1}{2}\left\langle q, G[q]\right\rangle. \tag{3.8}$$

This implies that H is indeed the kinetic energy of the fluid.

3.1. Problem 1: Unconstrained extremization of E

In the first problem the radius R, β and μ are fixed but the total circulation Ω is unconstrained except by the bound M in $V'(R)$. The first problem concerns the unconstrained extremization of E and is given by (3.9) - (3.10): Extremize

$$E[q(z), \beta, \mu] = H + \frac{\mu}{\beta}\Gamma \text{ for fixed } \beta > 0 \text{ and } \mu > 0 \tag{3.9}$$

in

$$V'(R) = \left\{q \in L^2(D_R) \mid 0 < q(z) \leqslant M \text{ a.e.}\right\} \tag{3.10}$$

where the radius R of the disk D_R is fixed but arbitrary. There are two cases to be considered: (a) when $\ln R \leqslant \frac{1}{4}$ and (b) when $\ln R > \frac{1}{4}$. The upper bound M on $q(z)$ in $V'(R)$ is needed to bound E from below in case (b). The positivity of $q(z)$ a.e. in D_R in the definition (3.10) not only excludes the trivial function $q(z) \equiv 0$, but more importantly, it fixes the support of $q(z)$ to be precisely D_R and not any smaller disks with radii $R' < R$. This is important for the reason that in problem 1, we are enforcing the constraint that the support of q is precisely D_R while the total circulation is free.

We will prove the following results:

Proposition 1. (a) *E is bounded below by 0 when* $\ln R \leqslant \frac{1}{4}$. (b) *When* $\ln R > \frac{1}{4}$, *the extremal* q_* *is a saddle point and the following relation holds between the total circulation* Ω, β, μ and R :

$$\frac{R^2}{|4\ln R - 1|} = \frac{\Omega\beta}{4\pi\mu}. \tag{3.11}$$

Proof: The Euler–Lagrange equation for this problem is given by

$$G[q_*](x) = -\frac{\mu}{\beta}|x|^2 \text{ for } q_* \in V'(R). \tag{3.12}$$

The only solution of (3.12) is the constant function

$$q_*(x) = q_0 I_R(x),\ \ q_0 > 0$$

where $I_R(x)$ is the characteristic function of the disk D_R and q_0 is a positive constant. Substituting this solution into (3.12) implies that

$$\begin{aligned} 0 &= \left\langle q_*, G[q_*] + \frac{\mu}{\beta}|x|^2 \right\rangle = \\ &= q_0 \int_{D_R} G[q_0 I_R]dx + \frac{\mu q_0}{\beta}\int_{D_R} |x|^2 dx. \end{aligned} \tag{3.13}$$

After evaluating the integrals, we get

$$\frac{\pi R^4 q_0^2}{8}(1 - 4\ln R) + \frac{\pi R^4 \mu q_0}{2\beta} = 0 \tag{3.14}$$

whose solutions are

$$\begin{aligned} &(a)\ q_0 = 0 \text{ for all } R > 0, \\ &(b)\ q_0 = \frac{4\mu}{\beta(4\ln R - 1)} > 0 \text{ when } \ln R > \frac{1}{4}. \end{aligned} \tag{3.15}$$

Since for $q_0 = 0$, the function $q_*(x) = q_0 I_R(x)$ does not satisfy the Euler-Lagrange equation (3.12) and moreover the trivial function $q_*(x) \equiv 0$ is not in $V'(R)$, the only interesting stationary point above is given by (3.15).

A simple calculation in terms of

$$q(x) = q_0 I_R(x) + q'(x) \tag{3.16}$$

where

$$\int_{D_R} q' dx = 0, \tag{3.17}$$

shows that

$$\begin{aligned} E[q] &= H[q] + \frac{\mu}{\beta}\Gamma[q] = \\ &= E_0(q_0) + \langle q_0 I_R(x), G[q'] \rangle + E_1[q'], \\ E_0(q_0) &= \frac{\pi R^4 q_0^2}{16}(1 - 4\ln R) + \frac{\mu\pi R^4 q_0}{2\beta}, \\ E_1[q'] &= \frac{1}{2}\langle q', G[q'] \rangle + \frac{\mu}{\beta}\langle q', |x|^2 \rangle. \end{aligned} \tag{3.18}$$

Using the basis B in (3.5), (3.6), and (3.17) to expand q' in (3.16), we get

$$\langle q_0 I_R(x), G[q'] \rangle = q_0 \sum q'_m \lambda_m \langle I_R, \psi_m \rangle = 0.$$

$E_1[q']$ is strictly convex in q' by Lemma 2a. Thus when (a) $4\ln R \leqslant 1$, $E[q]$ is strictly convex in $V'(R)$ which implies that

$$E[q] > E_0(0) = 0 \text{ for all } q \in V'(R).$$

The trivial function $q \equiv 0$ is the unique minimizer of E if we close $V'(R)$ by including the trivial function.

When (b) $4\ln R > 1$, the concavity of $E_0(q_0)$ in q_0 implies that the extremal q_0 in (3.15) is a saddle point. The lower bound of $E[q]$ is achieved at the boundary of $V'(R)$, i.e., when $q(x) = MI_R(x)$.

Substituting the saddle point q_0 (3.15) into the definition of total circulation

$$\int_{D_R} q dx = \Omega$$

yields

$$\frac{4\mu\pi R^2}{\beta(4\ln R - 1)} = \Omega$$

which proves (3.11). This completes the proof of the proposition.

Remark 1. As it stands $V'(R)$ is convex but not closed. By adding the trivial function $q \equiv 0$ to $V'(R)$, we get a closed and convex set $\bar{V}'(R)$. It follows from lemma 2a and lemma 3 that E achieves its minimum in $\bar{V}'(R)$ at the unique minimizer $q \equiv 0$.

Remark 2. The minimizer $q(x) = MI_R(x)$ of $E[q]$ in $V'(R)$ is uninteresting because it is essentially an artifact of the upper bound M on q in $V'(R)$, without which, $E[q]$ is not bounded below. The saddle point cannot be located by a Monte-Carlo method which will instead select the global minimizer $q(x) = MI_R(x)$. The existence of a saddle point and the unboundedness above and below of $E[q]$ when $\ln R > 1/4$ implies that the augmented functional $E[q]$ makes for a poor choice of the argument in a Gibbs canonical probability measure $P(q) = Z^{-1}e^{-\beta E[q]}$ unless the total circulation is fixed explicitly or in a microcanonical way. It should be noted that for a finite vortex gas, the fixed total circulation constraint is implicit in the Onsager partition function.

The proof of the following lemma is the same as that of Lemma 2 in the next section and will be omitted.

Lemma 2a. *The functional* $E_1[q']$ *defined by* (3.16) *and* (3.18) *is strictly convex.*

3.2. Problem 2: The variable radius variational problem

For comparisons with the results of our Monte-Carlo simulations in which a fixed number N of point vortices of strength one are taken to equilibrium with energy and angular momentum reservoirs of fixed inverse temperature $\beta > 0$ and chemical potential $\mu > 0$, we formulate the following variational problem in which the total circulation $\Omega = N > 0$ is fixed and the supports of the vorticity density distributions q are allowed to vary over disks of finite radii R :

$$\min_{V(\Omega)} E[q; \beta, \mu] \text{ where} \tag{3.19}$$

$$V(\Omega) = \left\{ \begin{matrix} 0 \leqslant q \in L^2(R^2) \text{ such that supp } (q) = D_R \text{ for } 0 < R < \infty \\ \text{and } \int_{D_R} q dx = \Omega > 0 \end{matrix} \right\}.$$

The main result in this paper is theorem 4. Since the Euler-Lagrange equations are complicated for variational problems defined on a variable domain, we prefer to use a more direct method in the proof below. We will first prove a strict convexity lemma used in this section and below.

Lemma 2. *The energy functional* $E[w;\eta=\frac{\mu}{\beta}]$ *is strictly convex in* w *on*

$$V_R(\Omega)=\{q\in V(\Omega) \text{ with the same supp, } D\}\,.$$

Proof: Let $\lambda\in(0,1)$ and $w_1\neq w_2$ be in $V_R(\Omega)$. Then

$$w=\lambda w_1+(1-\lambda)w_2\in V_R(\Omega).$$

The strict convexity of $E[w;\eta]$ follows from that of

$$E'[w;\eta]=E[w;\eta]-E_0[\Omega,\eta]$$

which differs from E by a constant.

We want to show that

$$E'[w;\eta]=E'[\lambda w_1+(1-\lambda)w_2;\eta]<\lambda E'[w_1;\eta]+(1-\lambda)E'[w_2;\eta].$$

The left hand side,

$$\begin{aligned}&E'[\lambda w_1+(1-\lambda)w_2;\eta]=\\ =&\int_{R^2}dx\int_{R^2}dx'(\lambda w_1+(1-\lambda)w_2)(x)(\lambda w_1+(1-\lambda)w_2)(x')g_{\eta,\Omega}(x,x')-E_0[\Omega,\eta]=\\ =&\lambda^2E'[w_1;\eta]+(1-\lambda)^2E'[w_2;\eta]+2\lambda(1-\lambda)E'[w_1,w_2],\end{aligned}$$

where

$$E'[w_1,w_2]=\int_{R^2}dx\int_{R^2}dx'w_1(x)w_2(x')g_{\eta,\Omega}(x,x')-E_0[\Omega,\eta]$$

and

$$g_{\eta,\Omega}(x,x')=\frac{1}{4\pi}\log|x-x'|^{-1}+\frac{\eta}{2\Omega}|x|^2+\frac{\eta}{2\Omega}|x'|^2\geqslant c'>-\infty \tag{3.20}$$

on $D\times D$. We need to prove that

$$\begin{aligned}&\lambda^2E'[w_1;\eta]+(1-\lambda)^2E'[w_2;\eta]+2\lambda(1-\lambda)E'[w_1,w_2]<\\ &<\lambda E'[w_1;\eta]+(1-\lambda)E'[w_2;\eta]\end{aligned}$$

which is equivalent to

$$\lambda(1-\lambda)\left\{\big(E'[w_1;\eta]-E'[w_1,w_2]\big)+\big(E'[w_2;\eta]-E'[w_1,w_2]\big)\right\}>0.$$

The left hand side is equal to

$$\begin{aligned}&\lambda(1-\lambda)\left\{\begin{array}{c}\int_{R^2}dx\int_{R^2}dx'w_1(x)w_1(x')g_{\eta,\Omega}(x,x')-\\ -\int_{R^2}dx\int_{R^2}dx'w_1(x)w_2(x')g_{\eta,\Omega}(x,x')\\ +\int_{R^2}dx\int_{R^2}dx'w_2(x)w_2(x')g_{\eta,\Omega}(x,x')-\\ -\int_{R^2}dx\int_{R^2}dx'w_1(x)w_2(x')g_{\eta,\Omega}(x,x')\end{array}\right\}=\\ &=\lambda(1-\lambda)\left\{\begin{array}{c}\int_{R^2}dx\int_{R^2}dx'w_1(x)(w_1(x')-w_2(x'))g_{\eta,\Omega}(x,x')-\\ -\int_{R^2}dx\int_{R^2}dx'w_2(x)(w_1(x')-w_2(x'))g_{\eta,\Omega}(x,x')\end{array}\right\}=\\ &=\lambda(1-\lambda)E[w_1-w_2;\eta]\end{aligned}$$

after using the symmetry and the bilinearity of the quadratic form.

It remains to show that

$$E[w_1-w_2;\eta]>0$$

for $w_1 \neq w_2$ in $V_R(\Omega)$. Since both w_1 and w_2 are in $V_R(\Omega)$,

$$\int_{R^2} (w_1 - w_2)(x)dx = 0;$$

thus, $w_1 - w_2$ is no longer in $V_R(\Omega)$ and is not necessarily single-signed. Since

$$\begin{aligned}\int_{R^2} dx \int_{R^2} dx'(w_1 - w_2)(x)(w_1 - w_2)(x')\frac{\eta}{2\Omega}|x|^2 = \\ = \frac{\eta}{2\Omega}\int_{R^2} dx|x|^2(w_1 - w_2)(x)\int_{R^2} dx'(w_1 - w_2)(x') = 0,\end{aligned}$$

and similarly,

$$\int_{R^2} dx \int_{R^2} dx'(w_1 - w_2)(x)(w_1 - w_2)(x')\frac{\eta}{2\Omega}|x'|^2 = 0,$$

we get

$$\begin{aligned}E[w_1 - w_2; \eta] &= \int_{R^2} dx \int_{R^2} dx'(w_1 - w_2)(x)(w_1 - w_2)(x')g_{\eta,\Omega}(x, x') = \\ &= \int_{R^2} dx \int_{R^2} dx' \left[\begin{array}{c} (w_1 - w_2)(x)(w_1 - w_2)(x') \\ \times \left(\frac{1}{4\pi}\log|x - x'|^{-1} + \frac{\eta}{2\Omega}|x|^2 + \frac{\eta}{2\Omega}|x'|^2\right) \end{array} \right] = \\ &= \frac{1}{4\pi}\int_D dx \int_D dx'(w_1 - w_2)(x)(w_1 - w_2)(x')\log|x - x'|^{-1}.\end{aligned}$$

The last expression can be written in the form $\frac{1}{2}\langle q, G[q]\rangle$ where G is a compact self-adjoint operator according to (3.3) and (3.4). By the Spectral theorem for compact self-adjoint operators, there is an orthonormal basis of $L^2(D)$ made up of eigenfunctions ψ_j of G (with eigenvalue λ_j) which diagonalizes G :

$$\begin{aligned}G[q](x) &= \frac{1}{2\pi}\int_D dx' \left(\sum_j q_j\psi_j(x')\right)\log|x - x'|^{-1} = \\ &= \sum_j q_j\lambda_j\psi_j(x).\end{aligned} \tag{3.21}$$

The eigenvalues $\lambda_j > 0$ tends to 0 as j tend to infinity since $L^2(D)$ is infinite-dimensional. Thus, for q in $L^2(D)$,

$$\begin{aligned}\langle q, G[q]\rangle &= \left\langle \sum_j q_j\psi_j(x), G\left(\sum_k q_k\psi_k(x')\right)\right\rangle = \\ &= \sum_j q_j^2\lambda_j \geqslant 0.\end{aligned} \tag{3.22}$$

Since $q = w_1 - w_2 \neq 0$, at least one of the coefficients $q_j(w_1 - w_2) \neq 0$ which implies that

$$\langle w_1 - w_2, G(w_1 - w_2)\rangle = \sum_j q_j^2(w_1 - w_2)\lambda_j > 0.$$

This completes the proof of strict convexity of $E[w; \eta]$ on $V_R(\Omega)$.

Lemma 3. *If a functional $E[q]$ is strictly convex on a convex set V of functions q with support in a compact subset $D \subset R^2$, then the minimizer of E* in *V is unique and radially symmetric.*

Proof. By Prop 1.1 on page 35 in [9], $\min E$, the set of minimizers of E in V is a closed convex set which is non-empty by convexity of E. Strict convexity of E in $\min E$ implies uniqueness of the minimizer of E in V. Suppose that there are two minimizers q_1 and q_2 of E in V. Thus, $\frac{1}{2}(q_1+q_2)$ is also a minimizer of E in V which contradicts the strict convexity of E since

$$\alpha = E\left(\frac{1}{2}(q_1+q_2)\right) < \frac{1}{2}E(q_1) + \frac{1}{2}E(q_2) = \alpha.$$

Uniqueness of the minimizer implies that the minimizer q is radial. Suppose q is not radial; then a rotation q_θ of q is another element in the set of minimizers of E in V. This contradicts the property of uniqueness and completes the proof of lemma 3. ∎

Theorem 4. *The variational problem of the minimization of augmented energy functional*

$$E[q;\beta,\mu] = H(q) + \frac{\mu}{\beta}\Gamma(q), \;\; \beta > 0, \;\; \mu > 0$$

in the set $V(\Omega)$ of square-integrable, a.e. bounded and single-signed vorticity distributions $q(r,\theta)$ of fixed total circulation $\Omega > 0$ with support equal to disks D_R of any finite radius $R \in (0,\infty)$ has a solution, that is, E takes its minimum in $V(\Omega)$, at the unique radial minimizer

$$q_m = \frac{4\mu}{\beta} I_R(r) \tag{3.23}$$

where the radius is given by

$$R = \sqrt{\frac{\Omega\beta}{4\pi\mu}}. \tag{3.24}$$

Proof. Consider the one parameter family of orthonormal bases $\{\psi_j^{(R)}\}$ parametrized by R in $(0,\infty)$, corresponding to the square-integrable classes $L^2(D_R)$ with R ranging over the same values. These bases come from the Spectral Theorem applied separately to each $L^2(D_R)$ with $R < \infty$. Each vorticity distribution $q^{(R)}(x)$ in $L^2(D_R)$ can be written in the form

$$q^{(R)}(x) = q_0^{(R)} I_R(x) + q'^{(R)}(x). \tag{3.25}$$

The fixed total circulation Ω completely determines the first term $q_0^{(R)} I_R(x)$ and the total circulation of $q'^{(R)}(x)$ is zero.

Expanding $E[q^{(R)}]$ yields

$$E[q^{(R)}] = E_0(R) + \left\langle q_0^{(R)} I_R(x), G[q'] \right\rangle + E_1[q'^{(R)}], \tag{3.26}$$

$$E_0(R) = \frac{\pi R^4 \left[q_0^{(R)}\right]^2}{16}(1 - 4\ln R) + \frac{\mu\pi R^4 q_0^{(R)}}{2\beta} =$$

$$= \frac{\Omega^2}{16\pi}(1 - 4\ln R) + \frac{\mu\Omega R^2}{2\beta}, \tag{3.27}$$

$$E_1[q'^{(R)}] = \frac{1}{2}\left\langle q'^{(R)}, G[q'^{(R)}]\right\rangle + \frac{\mu}{\beta}\left\langle q'^{(R)}, |x|^2\right\rangle,$$

where on expanding $q_0^{(R)} I_R(x)$ and $q'^{(R)}$ in terms of the orthonormal basis B for $L^2(D_R)$ in (3.5),

$$\left\langle q_0^{(R)} I_R(x) I_R(x), G[q'] \right\rangle = \left\langle q_0^{(R)} I_R(x), G[\sum q'_m \psi_m] \right\rangle =$$

$$= q_0^{(R)} \sum q'_n \lambda_n \left\langle I_R(x), \psi_m \right\rangle = 0. \tag{3.28}$$

It follows from (3.27) that $E_0(R)$ is strictly convex in $R \in (0, \infty)$. For fixed R, the remainder term $E_1[q^{'(R)}]$ is also strictly convex in $q'^{(R)} \in L^2(D_R)$ such that $\int_{D_R} q'^{(R)} dx = 0$, by the same proof as in Lemma 2. Thus by (3.28), for R varying over all finite positive values, $E[q^{(R)}]$ is the sum of two parts $E_0(R)$ and $E_1[q'^{(R)}]$ which are separately, strictly convex in their respective arguments, R and $q'^{(R)}$. This implies that $E[q^{(R)}]$ is strictly convex in $V(\Omega)$.

In order to minimize $E[q^{(R)}]$, it follows from Lemma 2 that the minimum of $E_1[q'^{(R)}]$ over $q'^{(R)} \in L^2(D_R)$ for fixed R, such that $\int_{D_R} q'^{(R)} dx = 0$, is achieved by the unique minimizer $q'^{(R)}(x) \equiv 0$. Thus, $E[q^{(R)}]$ restricted to the one parameter family

$$q_0^{(R)}(x) = \frac{\Omega}{\pi R^2} I_R(x)$$

is given by

$$E_0(R) = \frac{\Omega^2}{16\pi}(1 - 4\ln R) + \frac{\mu \Omega R^2}{2\beta} \text{ with fixed } \Omega > 0, \beta > 0, \mu > 0. \tag{3.29}$$

Setting the derivative $\frac{d}{dR} E_0(R) = 0$ gives the scaling relation (3.24) and the form of the minimizer (3.23). The uniqueness of (3.23) follows from the strict convexity of $E[q^{(R)}]$ on the convex set $V(\Omega)$ in (3.19) and lemma 2. This completes the proof of theorem 4.

Remark 3. The form of (3.29) implies that relaxing the total circulation constraint in variational problem 2 leads to an uninteresting problem where the global minimizer is the trivial function $q = 0$ of zero total circulation.

4. Monte-Carlo simulations and discussion

In this section, we present some results from numerical simulations of the finite vortex gas, based on the Metropolis algorithm [20], and compare them with the rigorous scaling relation in theorem 4. The Monte-Carlo empirical results confirm the rigorous results in this paper. The variational models in this paper, which are derived from 2D Euler dynamics and are constructed to depend only on the ratio β/μ, predict properties similar to low and moderate temperatures Monte-Carlo simulations on a finite vortex gas.

Another series of lattice simulations is being carried out to compare with the scaling relationship (3.11) where the support of the vorticity density functions is a fixed disk and the total circulation is free to take any positive value. While the numerical results are not presented here, we have strong evidence for the dichotomy stated in proposition 1.

In the current series of Monte-Carlo experiments which suggested the low temperature formula (2.1) in the first place, we used N point vortices of unit strength, with N ranging from 50 to 1000 in the simulations. These point vortices were allowed to interact canonically with the energy and angular momentum or moment of vorticity reservoir determined by moderate to high β and μ respectively. They interact through a change in their positions. No constraints were imposed on the positions of the vortices except the angular momentum Γ_N.

After the vortices have equilibrated, the distance of the furthest vortex from the origin, $R_{\max}$ were measured. Figure 1 shows the typical vortex distribution after it has equilibrated. The vortices are well spread out inside a bounding circle. Figure 2 shows the linear relationship between $R_{\max}^2$ and N. Furthermore, the gradient of the plots fits the result proved in theorem 4. To help us interpret these numerical results in comparison to simulations at very high temperatures, we follow with a brief discussion on macro-states and micro-states.

For each β and μ, the Metropolis algorithm generates an ensemble of microstates $\vec{z} = (z_1, ..., z_N)$; with the probability for each microstate being $\exp(-E_N(\vec{z}, \beta, \mu))/Z_N$ where $Z_N = \int_{R^2} dz_1 ... \int_{R^2} dz_N \exp(-E_N(\vec{z}, \beta, \mu))$ is the normalization factor. In doing statistical experiments,

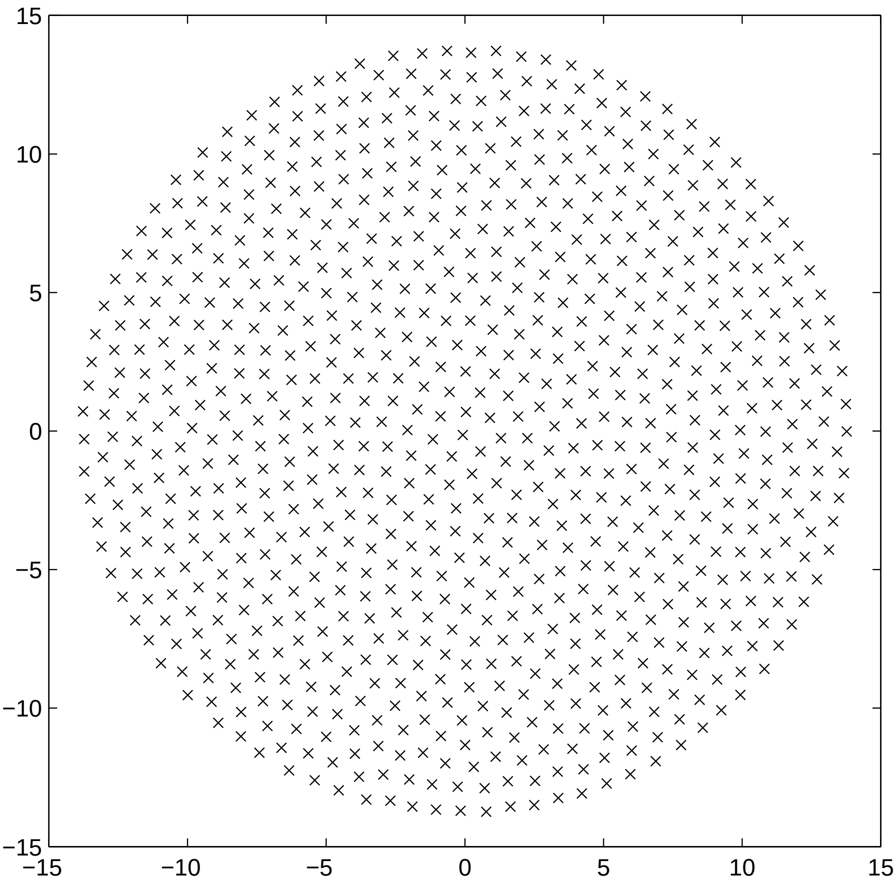

Fig. 1. Equilibrated point vortex distribution with $N = 800$ for $\beta = 2000\pi$ and $\mu = 2000$.

one would normally measure the dynamical variable for a macro-state by taking weighted-averages over all the micro-states:

$$\langle Q \rangle = \frac{1}{Z_N} \int_{R^2} dz_1 ... \int_{R^2} dz_N \; Q \exp(-E_N(\vec{z}, \beta, \mu))$$

where Q is any quantity associated with the macro-state. At large values of β, μ, one expects the microstates at or very near the minimum of E_N to dominate the integral for the average. In the usual statistical mechanics models, one cannot expect the same to be true in general.

In fact, high temperature simulations of the finite vortex gas clearly points to Gaussian like radial vorticity profiles, which are not ground states of the energy functional $E_\Omega[q]$. Numerical experiments on the full mean field pdes concur [32]. Only at low and moderate temperatures do the numerical experiments on the finite vortex gas support the above expectation that the macro-state is effectively identical to the ground state for large N. As such, the equality between the radius R of the support of the ground state in (3.24) and the expected value of the distance of the furthest vortex from the origin in (2.1), has been confirmed to an extremely high accuracy by the Monte-Carlo results reported here.

In conclusion, the rigorous variational approach in this paper yields ground states or global minimizers of $E[q]$, which are close to the most probable macrostate of the vortex gas at the low temperatures limit of the mean field vortex scaling theory. In view of the fact that the planar vortex gas is representative of many 2D and 2.5D statistical mechanics models for geophysical flows, the Principle of Minimum Energy is expected to provide a useful method for predicting the statistical properties of these models in a wide range of low to moderate temperatures.

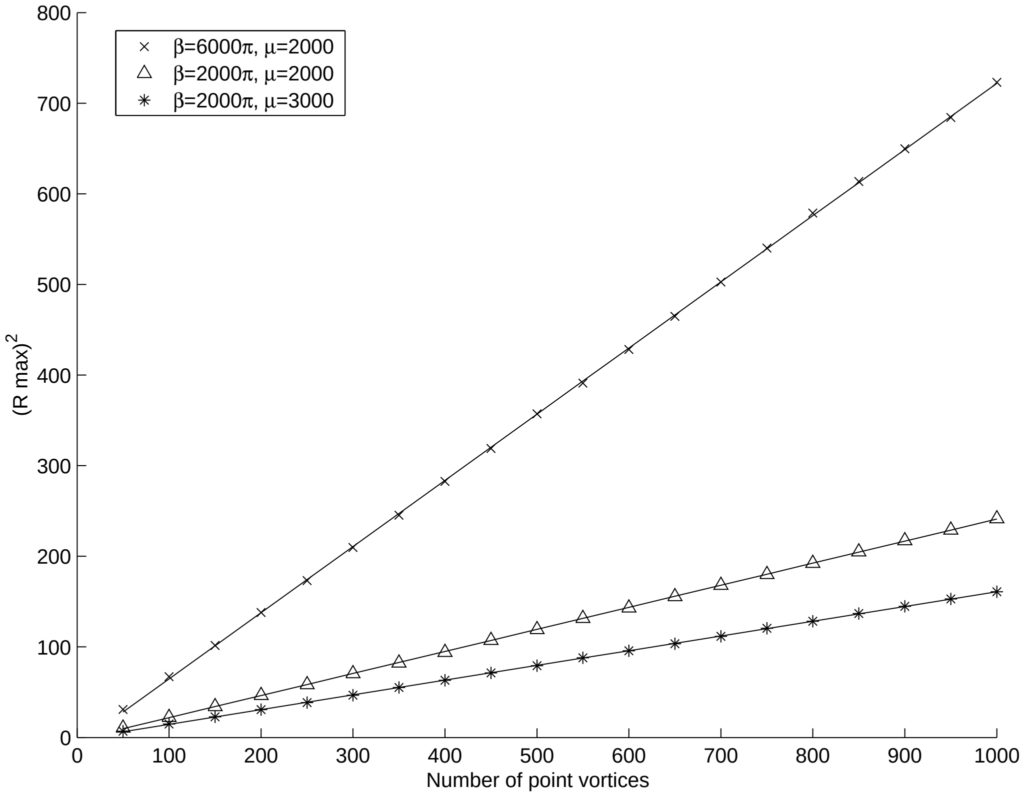

Fig. 2. Relationship between the distance of the furthest point vortex from the origin with the total number of vortices of unit circulation.

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