# CHAPTER 4.3 : CONTINUOUS CASE

Integral Transform: Suppose that X is continuously distributed and h(x) is a nice function; We expect h(X) = Y to be a random variable, and its density $g_Y(y)$ is :

First calculate its distribution function $G_Y(a) = P(Y < a) = P(h(X) < a)$

For increasing h(x), we can find the inverse $X < h^{-1}(a)$, and then
$G_Y(a) = F_X(h^{-1}(a))$

For decreasing h(x), again we can rewrite the event $\{h(X) < a\}$ as $\{X > h^{-1}(a)\}$;
Then $G_Y(a) = 1 - F_X(h^{-1}(a))$

Finally, differentiate to get: $g_Y(y) = d/dy\, G_Y = f_X(h^{-1}(y))\ |\, d/dy\, h^{-1}(y)\, |$

# APPLICATIONS TO NON MONOTONE CASE

Consider the rv X which is uniformly distributed in (-1, 2) or X is U(-1,2):

(a) Calculate density function, $f_X(x) = 1/3$ Ex

(b) Calculate the density function of $Y = X^2$ : $g_Y(y) = ??$

Ans: Note that the range of Y is (0, 4); then $G_Y(a) = P(Y < a) = P(\{X^2 < a\})$;

For $0 < a < 1$, $\{X^2 < a\} = \{- \text{sqrt}(a) < X < \text{sqrt}(a)\}$ and
$P(\{- \text{sqrt}(a) < X < \text{sqrt}(a)\}) = 2\text{sqrt}(a) / 3$. Ex

For $1 < a < 4$, $\{X^2 < a\} = \{X < \text{sqrt}(a)\}$ and $P(\{X < \text{sqrt}(a)\}) = (\text{sqrt}(a) + 1)/3$. Ex

Then $g_Y(y) = d/dy\, G_Y(a) = 1/(3\text{sqrt}(y))$ for $0 < y < 1$. Ex. Calculate it for $1 < y < 4$??

# INTEGRAL TRANSFORM

Theorem: Let X be a continuous rv with distribution function $F(a) = F_X(a) = P(X < a)$. Then $Y = F(X)$ is uniformly distributed U(0,1), I.e., $g_Y(y) = 1$ if y is in (0,1), and $g_Y(y) = 0$ if y is not in (0,1),

Proof: $G_Y(b) = P(Y < b) = P(F(X) < b) = P(X < F^{-1}(b))$ if F monotone increasing. If F not strictly

increasing, that is, for $a < c$, $F(a) = F(c)$, then take $F^{-1}(b) = \sup \{x \mid F(x) < b\}$.

Then $G_Y(b) = P(X < F^{-1}(b)) = F(F^{-1}(b)) = b$. QED

Ex: Why is $G_Y(b) = b$ for all b in (0,1) indicative of a U(0,1) distribution?.

# MORE ON JOINTLY DISTRIBUTED RV: CHAP 4.4

Read example 4.19 upto and including theorem 4. 22:

Theorem: Let X and Y be independent rv's with density functions $f_X$ and $f_{Y.}$

Then, Z = X + Y has density

$f_Z(z) =$ convolution of $f_X$ and $f_{Y.}$

Proof: The joint density function of X and Y is $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$. Thus,

The distribution function of Z = X + Y is $F_Z(t) = P(\{Z < t\})$

= double integral over $\{X + Y < t\}$ [ $f_X(x)\, f_Y(y)$ ]

= integral over reals [$f_X(x)$ dx ] times integral from -inf to (t-x) [$f_Y(y)$ dy ]

= integral over reals [$f_X(x)\, F_Y(t\text{-}x)$ dx]

because integral from -inf to (t-x) [$f_Y(y)$ dy ] = $F_Y(t\text{-}x)$ by definition.

# CONTINUED

Theorem: Suppose X and Y are jointly distributed with joint density $f_{X,Y}(x,y)$. Let $u = u(x,y)$, $v = v(x,y)$ be invertible mapping from $R^2$ to $R^2$ . And let $U = u(X,Y)$, $V = v(X,Y)$ be jointly

distributed rv's with density $g_{U,V}(u,v)$. Then $g_{U,V}(u,v) = f_{X,Y}(\ x(u,v),\ y(u,v)\ )\ J(u,v)$

Where $J(u,v)$ is the Jacobian of the inverse mapping $x = x(u,v)$, $y = y(u,v)$ .

Proof:

Recall the change of variables formula from Multivar Calc: from definition, for arbitrary event A,

P({(U,V) in A}) = integral over set A [$g_{U,V}(u,v)$ dudv ] = P({(X,Y) in B = $(u,v)^{-1}$ (A)}) or

integral over set A [$g_{U,V}(u,v)$ dudv ] = integral over set B [$f_{X,Y}(x,y)$ dxdy ]

= integral over set A [ $f_{X,Y}(x(u,v),y(u,v))\ J(u,v)$ dudv ]. Since A arbitrary,

$g_{U,V}(u,v) = f_{X,Y}(\ x(u,v),\ y(u,v)\ )\ J(u,v)$