Hwk 1

Due Sept 17 5 pm in LMS

and for now email a copy

to grader

Late hwks are

deducted 15%.

Problems seminar will go

over Hwk 1 on

Mon Sept 13.

Corollary 1.6

If $P(B_i)=0$ for all values of i, then $P(\bigcup_{i=1}^{\infty} B_i)=0$.

Proof

Write $A_n=\bigcup_{i=1}^{n} B_i$. Then $P(A_n)\le \sum_{i=1}^{n} P(B_i)=0$ by Boole's inequality (1.1). As the events $\{A_n\}$ plainly satisfy the conditions of the theorem, we see that $P(A)=\lim P(A_n)=0$. But $A=\bigcup_{n=1}^{\infty} A_n=\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{n} B_i=\bigcup_{i=1}^{\infty} B_i$, which establishes the result. □

EXERCISES

1.1 Find an expression in terms of $P(A)$, $P(B)$ and $P(A\cap B)$ for the probability that exactly one of the events A, B occurs.

1.2 To win the championship, City must beat both Town and United. They have a 60 % chance of beating Town, a 70 % chance of beating United, and an 80 % chance of at least one victory. What is the chance they win the championship?

1.3 Use induction to prove Boole's inequality, as stated above in expression (1.1). Deduce that

$$P\left(\bigcap_{i=1}^{n} A_i\right)\ge 1-\sum_{i=1}^{n} P\left(A_i^c\right).$$

1.4 Show that, if $P(B_n)=1$ for $n=1,2,\ldots$, then $P(\bigcap_{n=1}^{\infty} B_n)=1$.

1.5 Use Theorem 1.5 to prove that, if $B_1\supset B_2\supset B_3\supset\cdots$, then $P(\bigcap_{n=1}^{\infty} B_n)=\lim P(B_n)$.

1.6 In the notation of Corollary 1.3, prove *Bonferroni's inequalities*, i.e.

$$S_1-S_2+\cdots-S_{2k}\le P\left(\bigcup_{i=1}^{n} A_i\right)\le S_1-S_2+\cdots+S_{2k-1}$$

for $k=1,2,3,\ldots$.

Corollary 1.6

If $P(B_i) = 0$ for all values of i, then $P(\bigcup_{i=1}^{\infty} B_i) = 0$.

Proof

Write $A_n = \bigcup_{i=1}^{n} B_i$. Then $P(A_n) \leq \sum_{i=1}^{n} P(B_i) = 0$ by Boole's inequality (1.1). As the events $\{A_n\}$ plainly satisfy the conditions of the theorem, we see that $P(A) = \lim P(A_n) = 0$. But $A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{\infty} B_i$, which establishes the result. □

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for $k = 1, 2, 3, \ldots$.

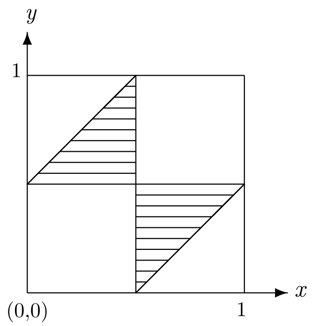

Figure 1.2 Where (x, y) must fall to enable a triangle to be formed

the circumference of the circle (just like choosing two random points on a stick) and join them. Another is to select two points P and Q at random in the interior of the circle, then extend the line PQ to a full chord. Or select just one point, P, at random in the interior, and draw the chord through P perpendicular to the radius joining P to the centre of the circle. Not only are these descriptions different in wording, they also lead to different answers to such questions as the average length of a chord, or the chance the chord is longer than the radius of the circle. These are just three of the many ways of giving a reasonable interpretation to the phrase "a random chord". That phrase, alone, is too vague to work with. That there are different answers is known as *Bertrand's paradox* (Bertrand 1889).

EXERCISES

1.7 Set up a probability space for tossing a fair coin three times. What is the chance all three tosses give the same result?

Find the flaw in the argument: "Plainly, at least two of the tosses are the same, both H or both T. As the third coin is equally likely to be H or T, there is a 50 % chance all three are alike."

1.8 Assume that the last two digits on a car number plate are equally likely to be any of the one hundred outcomes $\{00, 01, 02, \ldots, 98, 99\}$. Peter bets Paul, at even money, that at least two of the next n cars seen will have the same last two digits. Does $n = 16$ favour Peter or Paul? What value of n would make this a pretty fair bet?

1.9 Show that, for $0 < x < 1/2$,

$$x + \frac{x^2}{2} \le -\log(1-x) \le x + x^2.$$

We have seen that if we make K random selections among n equally likely objects, with replacement each time, the chance they are all different is $(n)_K/n^K = p_K$, say. Deduce that, if $K < n/2$, then

$$\frac{K(K-1)(2K-1)}{12n^2} \le -\frac{K(K-1)}{2n} - \log(p_K) \le \frac{K(K-1)(2K-1)}{6n^2}.$$

Hence show that, when n is large and we make about $\sqrt{-2n\log(p)}$ selections, the chance they are all different is close to p.

1.10 An examiner sets twelve problems, and tells the class that the exam will consist of six of them, selected at random. Gavin memorises the solutions to eight problems in the list, but cannot solve any of the others. What is the chance he gets four or more correct?

1.11 A poker hand consists of five cards from an ordinary deck of 52 cards. How many poker hands are

(a) Flushes (i.e. all five cards are from the same suit).

(b) Four of a Kind (e.g. four Tens, and a Queen).

(c) Two Pairs (e.g. two Kings, two Fours, and a Seven).

1.12 (a) Write $x(r) = (43)_r/(49)_r$. Explain why $x(r)$ is the probability that none of the numbers $\{1, 2, \ldots, r\}$ are in the winning combination in a given draw of a 6/49 Lottery.

(b) Let A_i be the event that the number i does not appear in the winning combination in n given Lottery draws. Find $P(A_i)$ and $P(A_i \cap A_j)$ when $i \neq j$ in terms of $x(1)$ and $x(2)$. Hence use Corollary 1.3 to get an expression for the probability that *some* number does not appear as a winning number among the next n draws. Use Bonferroni's inequalities (Exercise 1.6) to evaluate this when $n = 50$.

1.13 Let $P = (x, y)$ be a point chosen at random in the unit disc, centre $(0, 0)$ and radius 1. Describe the model of this experiment, and evaluate the probabilities that P is within 0.5 of the centre; that $y > 1/\sqrt{2}$; and that both $|x - y| < 1$ and $|x + y| < 1$.

1.14 A breakdown vehicle cruises along the straight road of unit length that links Newtown to Seaport; help for stranded motorists is also