Path Integral Monte Carlo Applied to Nearly Parallel Vortex Filaments

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Abstract

The Nearly Parallel Vortex Filament (NPVF) model has a wide range of applications from classical fluid turbulence to electron plasmas to superfluids and Bose-Einstein Condensates. Ceperley's Path Integral Monte Carlo (PIMC) algorithms provide an excellent tool for calculating the statistics of these systems in equilibrium, as Sen, Trivedi and Ceperley showed for flux lines of type-II superconductors by Sen et al. (2001). In this poster we discuss our application of PIMC to NPVF to study the melting of vortex crystals under a trapping potential when trapping intensity (e.g. angular velocity) increases, forcing higher vortex densities. We discover a phase transition in which control of the statistics of containment radius transfers from vortex interaction in the crystal phase to internal vortex fluctuations in the liquid phase. At the critical point entropy rather than internal energy becomes the main factor deciding the system state, consequently the radius becomes chaotic. This work represents the first numerical evidence for the kind of turbulence occuring at high rotation speed due to vortex tangling. We argue that the simplified, NPVF model is sufficient to make these assertions and that the full Biot-Savart law is unnecessary.

Introduction

Since the 1940's the point vortex model has been used as a major simplification of the Euler equations for inviscid fluid flows. In the statistical mechanics literature, they appear as points in a two-dimensional plane. However, adding a z-axis makes them perfectly straight, parallel, infinitely long vortex filaments. The advantage of the third dimension comes in when we begin to perturb the filaments. Rather than having perfectly straight filaments, we can make them nearly straight and, with random perturbations, represent local self-induction by Klein et al. (1995). This model assumes that filaments are periodic in the z-direction with period much longer than the inter-vortex distance and that this inter-vortex distance is much larger than the average vortex deviation from center line.

The Path Integral Monte Carlo (PIMC) method, first developed for simulations of bosons in a meshless space by Ceperley (1995), has been successfully applied to vortex filaments in high-T_c superconductors by Nordborg and Blatter (1998) and later by Sen et al. (2001). Its advantages in computational speed and its ability to simulate paths in a theoretically continuous space make it ideal for our purpose.

In this poster we discuss the results of PIMC simulations of NPVFs in the case where rotational speed and circulation are varied. Our results are two-fold: first we discuss our numerical confirmation of the containment radius formula derived variationally by Assad and Lim (2005) and secondly we discuss our numerical verification of a novel sort of turbulence in which high rotational speed results in vortex melting and a transition from a vortex crystal whose configuration is determined by internal energy to a vortex liquid, entropic in nature. This result is unique in that it is the first time such numerical simulations have been done.

Model and Method

A simplification of Navier-Stokes gives the following PDE as a model for NPVFs:

$$-i\partial_t\psi_j = \alpha\partial_{\sigma\sigma}\psi_j + \frac{1}{2}\sum_{j\neq k}\frac{\psi_j-\psi_k}{|\psi_j-\psi_k|^2},$$

where $\psi_j(\sigma,t)\in\mathbb{C}$ represents the position in the plane of the jth filament, the parameter σ is the position along the filment. Filaments are periodic with period L. This PDE model then gives a Hamiltonian that, for piecewise linear paths of M segments of length $\delta = L/M$ is:

$$H_N(M) = \alpha\sum_{j=0}^{M-1}\sum_{k=1}^{N}\frac{1}{2}\frac{|\psi_k(j+1)-\psi_k(j)|^2}{\delta} - \sum_{j=0}^{M-1}\sum_{i\neq k}^{N}\delta\log|\psi_i(j)-\psi_k(j)|,$$

where ψ_i are the positions of the filaments in the x-y plane, N is the number of filaments, and L is the length of each filament. We also have conserved angular momentum:

$$I_N(M) = \sum_{j=0}^{M-1}\sum_{k=1}^{N}\delta|\psi_k(j)|^2$$

Since our system interacts with a heat bath, the measure of probability of a state for our system is:

$$G_N(M) = \frac{1}{Z_N(M)}\exp(-\beta H_N(M) - \mu I_N(M))$$

where $v=\mu/\beta$ is the overall rotational velocity. Monte Carlo simulations using this model result in stationary states like Figures 1 and 2.

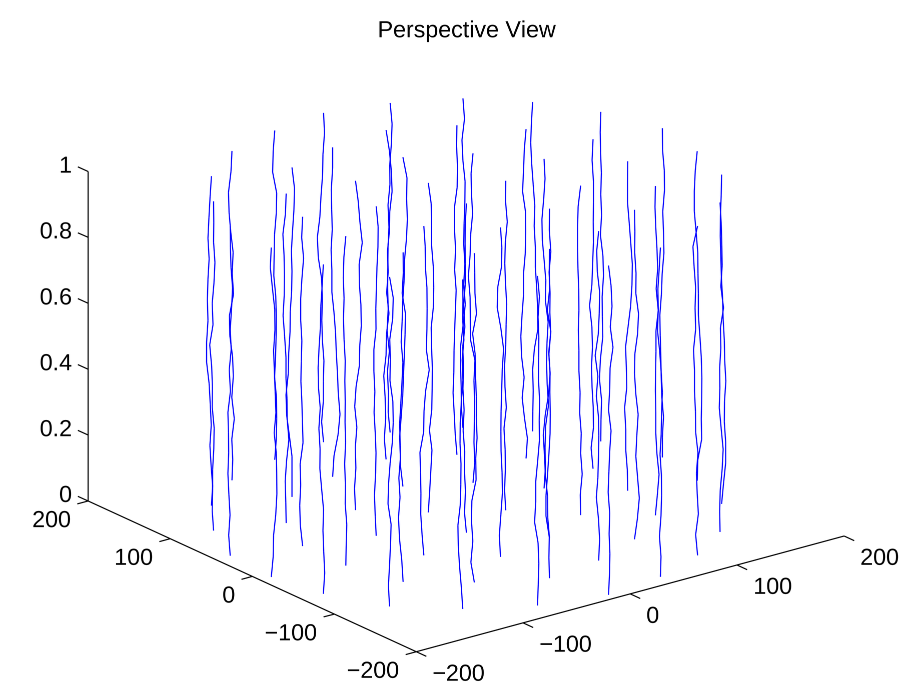

Figure 1. An example of the steady state. This result is reminiscent to the work of Tsubota et al. (2003) using the full Biot-Savart laws on a vortex tangle under rotation. However, that model relied on a cylindrical boundary to contain the flow and used a thermal counter flow to induce self-induction.

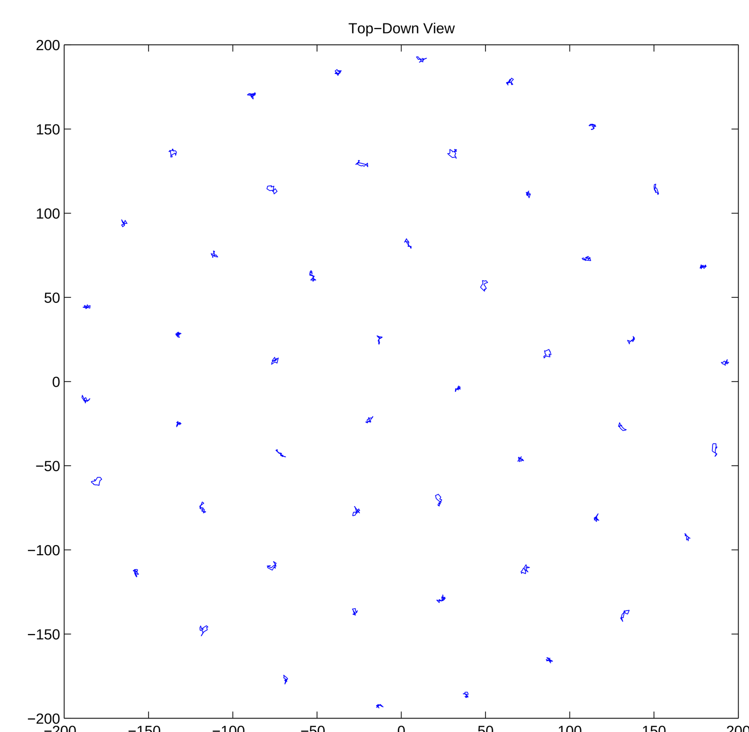

Figure 2. The top-down view from Figure 1. When projected onto the complex plane the filaments become random walks whose centers form a triangular lattice.

Monte Carlo Moves

Our Monte Carlo moves can be broken into two pieces. The first is the wholechain move in which a filament, selected with uniform randomness, is moved by a distance ϵ. This epsilon is a uniformly distributed random variable. We typically confine it to the range $[-1,1]$. This type of move allows the vortices to move to the correct points in the domain as a whole.

The second move decides the configuration of the vortex filament internally by doing many moves in series. The move starts with the two endpoints R_1 and R_N, ignoring the previous internal positions of beads, and chooses a bead, R_M, within a Gaussian box with mean $\bar{R}_M = \frac{R_1+R_N}{2}$ and variance $\sigma^2 = \frac{\delta}{\alpha\beta}$, where $\delta = L$. An example is shown in Figure 3. It can be shown that this distribution samples perfectly the case where there is no interaction.

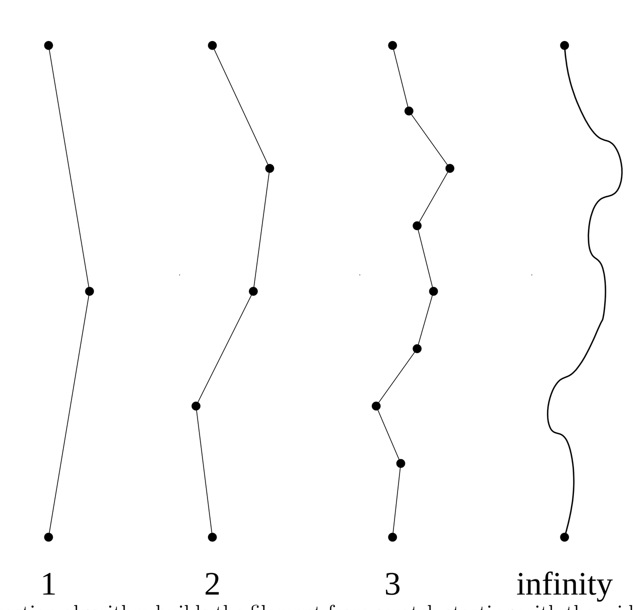

Figure 3. The bisection algorithm builds the filament from scratch starting with the middle point and adding points in between existing points. This figure shows an example of the first three levels and the infinite limit, a continuous curve.

Numerical Results for Varying Circulation

Our first finding is a repetition of Assad and Lim's findings which demonstrate the relationship between square containment radius, R, and the parameters β and μ and the circulation Ω, where α was allowed to remain fixed. We find near perfect agreement in the line slopes to Assad and Lim's formula for square containment radius $R^2 = \Omega\beta/(4\pi\mu)$. Note that the circulation of each vortex is equal to its period ($L=20$). Additionally, the Hamiltonian used in Assad and Lim (2005) is divided by π, which is not done Lions and Majda (2000). Therefore, the formula for our square containment radius is

$$R^2 = NL\beta/(4\pi^2\mu),$$

which gives the slope equal to $L\beta/(4\pi^2\mu)$ in agreement Figure 4.

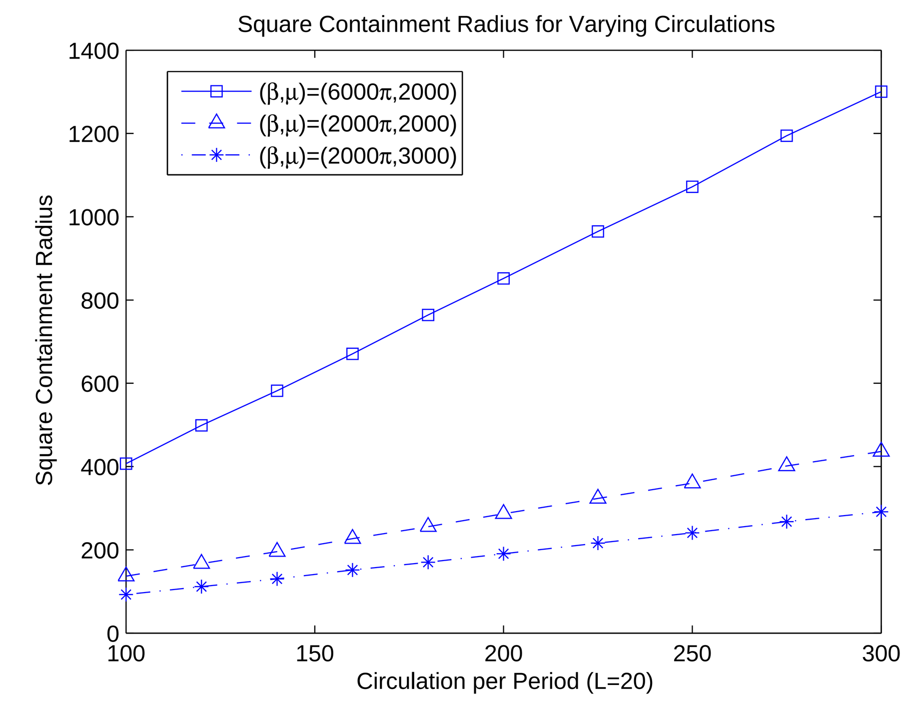

Figure 4. The square containment radius is linear w.r.t. circulation as expected from Assad and Lim (2005). The formula $R^2 = \Omega\beta/(4\pi\mu)$ appears valid.

The above result is of interest because it demonstrates a decoupling of the internal filament fluctuations and the overall statistics of the system at least for the primary statistics of the square containment radius. However, these experiments use very low temperature, meaning that the filaments are quite rigid. When β drops to low values (high temp.) the filaments begin to collide, violating a central assumption of the model that filaments remain a good distance apart (preferably well beyond the healing length.) When filaments do come close together they become entangled, and the interaction repulsion tears the filaments apart. At this point the containment radius statistics blow up as shown in Figure 6.

Numerical Results for Varying β

The model's blow up clearly results from the expansion of the circle that each vortex filament covers in the plane. For relatively low temperature system of vortices this "blob size" is nearly a point in comparison to intervortex spacing. It becomes important when vortex matter is hot. When β is varied we note a polynomial decline of blob size, visible in Figure 7. Since the relationship does not appear to change at $\beta = 10$ as the containment radius statistics did in Figure 6, we can conclude that the containment radius increased only because the "blobs" could no longer be contained–that is, the containment radius became, no longer a function of interaction, but a function of fluctuations. This phase transition is especially important because once fluctuations control containment radius, the radius of the axisymmetric flow (perhaps its most significant attribute) will vary chaotically rather than settling down to a fixed value. For a jet flow this would look like a cylinder of fluid with an aperiodically varying radius.

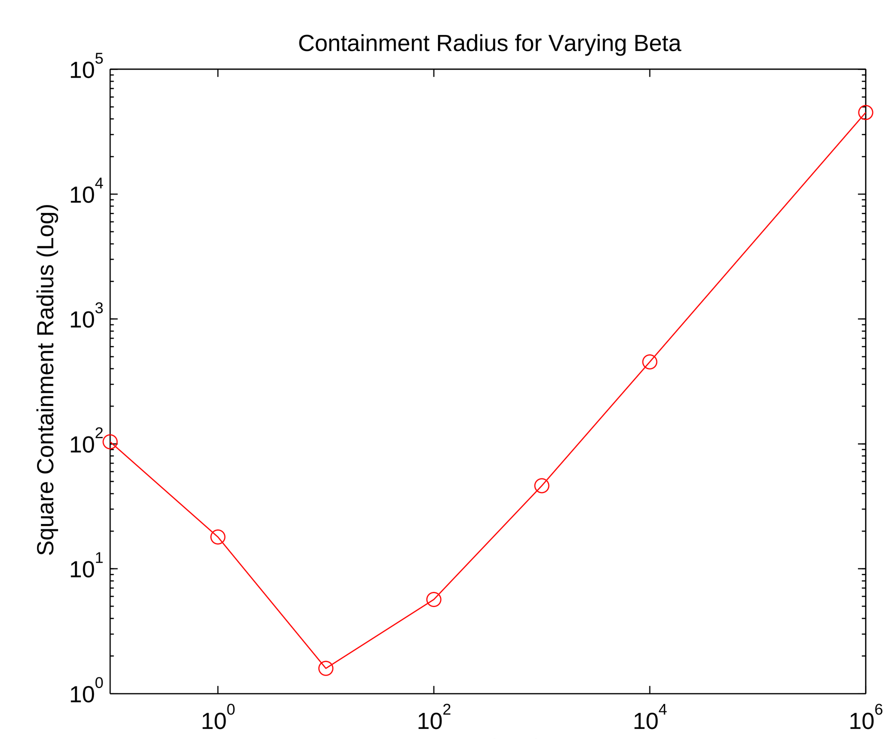

Figure 6. The model appears to break down when $\beta\sim 100$ ($\mu = 2000$) due to thermal fluctuations causing vortices to press agaist one another.

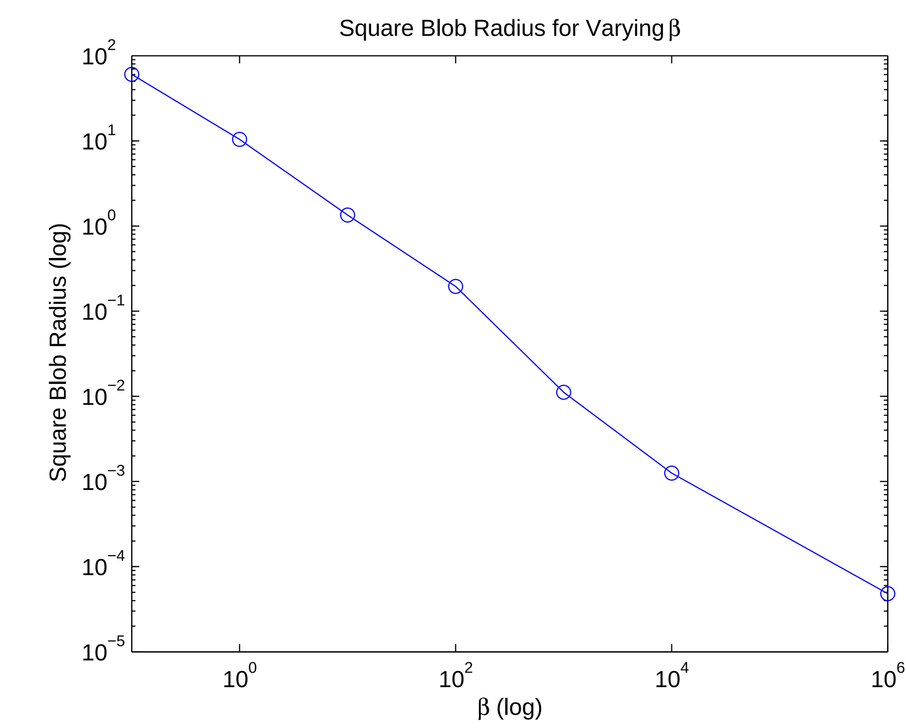

Figure 7. The nearly straight line indicates a polynomial relationship between blob size and the inverse temperature.

Conclusion

We have performed Monte Carlo experiments to attempt to confirm the decoupling of self-induction and vortex interaction. We have extended the results for containment radius statistics present in Assad and Lim (2005). These results help confirm Prantdl's assumption for slighly fluctuating jet flows. They may also be applied to electron plasmas and rotating Bose-Einstein Condensates. Many more experiments may be done to investigate the transition to chaos (only barely outlined here) at significantly large values of μ/β in the nearly parallel vortex filament model. This phase transition could be one of the most interesting aspects of the model since it is a clear departure from the point vortex paradigm. We remark that transitions to vortex tangles in superfluids are a form of turbulence as observed by Annett (2004). To study this transition is to violate the foundations of the model set down in Klein et al. (1995). However, it is only at the transition that the assumptions are truly invalidated. Therefore, the transition itself and its causes may be studied using this simple model. Once the transition is past we obtain a vortex tangle. Without a means of connecting or merging vortices, the model barely approximates a true tangle, and the full Gross-Pitaevskii (for superfluids) or Biot-Savart laws must be used instead. We propose that these experiments may be repeated for a fixed volume (making theoretical calculations easier) by increasing the number of vortices along with μ and keeping β fixed. The containment radius should hold steady until the vortices are too close together, after which the blobs take over and cause the radius to begin increasing.

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