Anomalous expansion and negative specific heat in quasi-2D plasmas

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Magnetic Nuclear Fusion

Magnetic nuclear fusion is one of the most promising avenues for renewable energy sources.

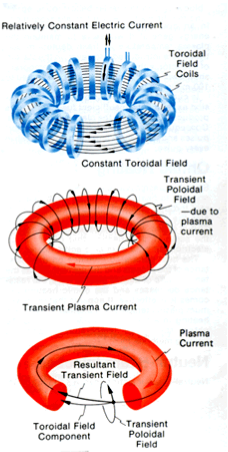


Figure: The tokamak is a toroidal ring with magnetic fields confining the plasma.

Magnetic Nuclear Fusion continued

$$ {}_1^2\mathrm{D} + {}_1^3\mathrm{T} \rightarrow {}_2^4\mathrm{He} + {}_0^1\mathrm{n} + 17.6\mathrm{MeV}. \quad (1) $$

- Plasma fuel (deuterium and tritium) is injected into the torus
- Electric current (about 3.5 MA) heat it to about 100 million K and align it.
- Ring currents confine the plasma to create sufficient central density for fusion.

Instabilities require explanation to be overcome

Confinement can only be sustained for a few seconds.

- Instabilities such as a sawtooth shaped cyclical drop in core temperature prevent sustained fusion.
- Expansion of the central density transports energy from the hot core to the cooler edge where it is lost.
- Instability and expansion are not well explained.

A statistical vortex model

Because of the high number of degrees of freedom in the plasma, we apply a statistical vortex model to explain:

- Anomalous expansion.
- Cyclical instability.

Electron Magnetohydrodynamic (EMH) Model

Electron plasmas.

The EMH model...

- has only one fluid, electrons (40,000 kps), ions (600 kps) are neutralizing background
- treats vorticity as generalized, $\mathbf{\Omega}(\mathbf{r}) = \nabla \times \mathbf{p}$, where

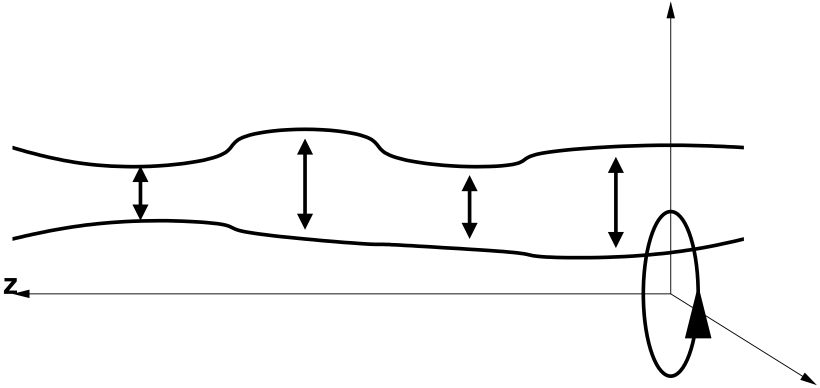
$$\mathbf{p} = m\mathbf{u} - e\mathbf{A}, \tag{2}$$

  is generalized momentum, m is electron mass, $-e$ is electron charge, and $\mathbf{A}$ is vector potential for the magnetic field.
- is combination of Navier-Stokes equations and Maxwell's equations
- implies equations of motion for *generalized* vorticity are the same as in neutral fluid model[1].

[1] [Uby et al.(1995)Uby, Isichenko, and Yankov]

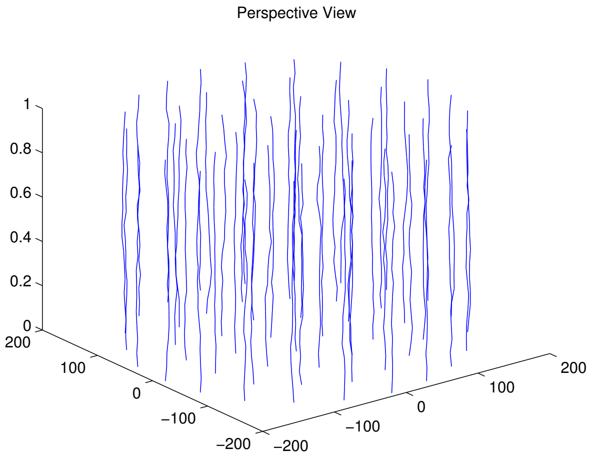
Nearly Parallel Vortex Filament Model

The interaction potential between two filaments, i and j, in any given plane parallel to the xy-plane is proportional to $\log(1/r_{ij})$, where r_{ij} is the distance between them.



Nearly Parallel Vortex Filaments

In large groups vortex filaments can be treated statistically and simulated on a computer.



Energy and Angular Momentum

Conserved quantities:

Energy

$$H_N = \alpha \int_0^1 d\tau \sum_{i=1}^{N} \frac{1}{2} \left| \frac{\partial \psi_i(\tau)}{\partial \tau} \right|^2 - \int_0^1 d\tau \sum_{i=1}^{N} \sum_{j>i}^{N} \log |\psi_i(\tau) - \psi_j(\tau)|, \quad (3)$$

where α is related to the velocity in the core, N is the number of filaments, and 1 is the period.

Angular Momentum

$$I_N = \sum_i \int_0^1 d\tau |\psi_i(\tau)|^2. \quad (4)$$

A state $s = \{\psi_1, \ldots, \psi_N\}$, where $\psi_j(\tau) = x_j(\tau) + iy_j(\tau)$, $\tau = z$, and $\psi_j(0) = \psi_j(1)$.

Canonical Probability Distribution

The canonical probability distribution applies to all classical physical systems in equilibrium.

For two conserved quantities,

$$P_c = Z_c^{-1} e^{-\beta H_N - \mu I_N}, \tag{5}$$

where β and μ are constants (μ/β is akin to pressure, $\beta = 1/T$ where T is temperature) and

$$Z_c = \int D\psi_1 \cdots D\psi_N \, e^{-\beta H_N - \mu I_N}, \tag{6}$$

is the normalizing *partition function*.

Prediction of Monte Carlo

The formula predicts the Monte Carlo Simulated expansion.

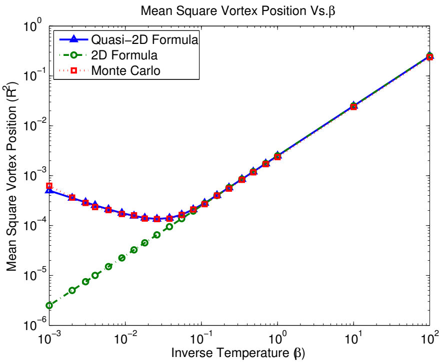


Figure: As β decreases beyond a threshold, expansion occurs in deviation from 2D point vortex results.

The rest of the story

Instability needs a microcanonical (fixed energy) ensemble.

- Assume a continuous density of vortex filaments at a particular cross-section, $g(\vec{x}, \vec{c})$ where $\vec{c} = d\psi/d\tau$.
- Define the entropy by Boltzmann,

$$S = -\int d^2x d^2c \, g \log g, \tag{7}$$

- Take the variation, $\delta S = 0$, with fixed energy, $E = \mathcal{T} + \mathcal{V}$.

$$\mathcal{T} = \frac{\alpha}{2} \int d^4(x,c) \, g c^2, \tag{8}$$

$$\begin{aligned} \mathcal{V} =& \frac{1}{2}\mu' \int d^4(x,c) \, g |\vec{x}|^2 - \\ & \frac{1}{\epsilon} \int d^4(x,c) \, d^4(x',c') \, g g' \log |\vec{x} - \vec{x}'|, \end{aligned} \tag{9}$$

Results

The variation results in an ODE:

$$\frac{d^2 v_1}{dz^2} + \frac{1}{z}\frac{dv_1}{dz} + e^{-v_1(z)+\mu z v_1'(z)} = 0, \quad v_1(0) = v_1'(0) = 0, \qquad (10)$$

- Energy,

$$E = \frac{\Lambda^2}{\epsilon}\left(\frac{z^2 e^{-v_1+\mu z v_1'}}{2(-z v_1')^2} - \frac{1}{(-z v_1')}\right), \qquad (11)$$

- Temperature,

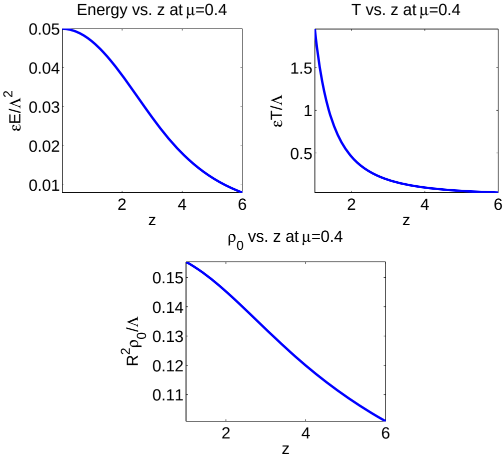
$$\frac{1}{T} = \beta = -\frac{\epsilon z v_1'(z)}{\Lambda}; \qquad (12)$$

- Core density,

$$\rho(0) = \frac{\epsilon z^2}{4\pi\beta R^2}, \qquad (13)$$

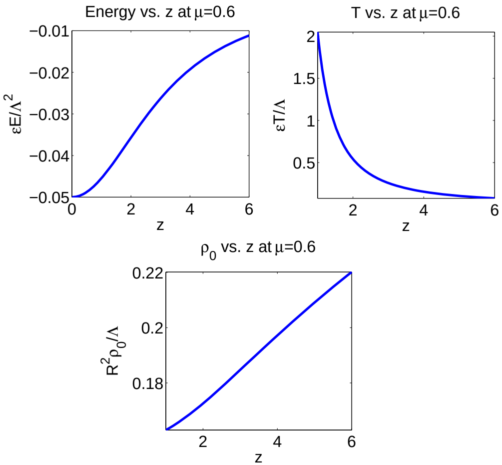
Low Confinement ($\mu < 0.5$)

At low confinement (unrealistic) the ensemble has positive specific heat.



High Confinement ($\mu > 0.5$)

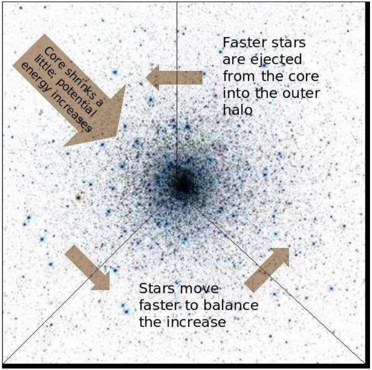
At high confinement (realistic) the ensemble has negative specific heat.



Gravothermal Catastrophe in Globular Clusters

Undergo "core collapse" with overall expansion

Statistical mechanics indicates that a gas of stars has negative specific heat (energy is inversely proportional to temperature), causing the system to be meta-stable. [2]



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[2] [Lynden-Bell and Wood(1968)]

Runaway Expansion Mechanism

Instead of a core collapse we have a core expansion.

- Slight expansion occurs and potential energy decreases.
- Kinetic energy increases in response.
- Increase in kinetic energy causes further expansion.
- Cycle continues until a more stable state is reached.

Conclusion

Because like vortices repel rather than attract, the runaway expansion is core collapse in reverse.

- The sawtooth *meta*stability seen in electron core temperature is the result of a runaway expansion.
- This mechanism also explains the anomalous expansion in the canonical ensemble.
- The forcing current creates a cycle.

D. Lynden-Bell and R. Wood.
The gravo-thermal catastrophe in isothermal spheres and the onset of red-giant structure for stellar systems.
Mon. Not. R. astr. Soc., 138:495–525, 1968.

L. Uby, M. B. Isichenko, and V. V. Yankov.
Vortex filament dynamics in plasmas and superconductors.
Phys. Rev. E, 52:932–939, 1995.