Notes on Incomplete One - period Markets and Fundamental Theorem of Asset Pricing

Chjan C. Lim

Let $(M_{m\times n}, S_0, P)$ be a one-period market where $n \geq m$, that is incomplete, i.e., $rank(M) = r < m$. Let $V \in R(M) = Range(M) = colspace(M)$, that is, it is hedgeable. So there is a hedge $X \in R^n$ such that

$$(*)\ MX = V. \tag{1}$$

Since

$$\begin{aligned} N + r' &= n, && (2)\\ N' + r &= m && (3) \end{aligned}$$

where $r = rank(M) = r' = rank(M^t)$, $N = \dim\ Nullspace(M)$, and $N' = \dim Nullspace(M^t)$,we get

$$N = n - r' \geq N' = m - r > 0. \tag{4}$$

Given that X ia a hedge for V, then $X' = X + k$ also solves (*) for any $k \in Nullspace(M)$. Thus,

Proposition 1 *Incomplete one-period markets have uncountably many hedges $X(V)$ for each hedgeable V.*

Consider the Replication method of pricing contingency claims V including call and put options: for a hedgeable $V \in Range(M)$, the Replication price of V is $V_0 = X^t S_0$ where X is a hedge for V. Since $X' = X + k$ is also a hedge for V, $V_0' = S_0^t X' = S_0^t(X + k) = S_0^t X + S_0^t k = V_0 + S_0^t k$. Thus,

Lemma 2 *For an incomplete one-period market, the Replication price of hedgeable claim V is well-defined (agrees for all hedges X' of V) if and only if $Nullspace(M) \perp S_0$.*

On the other hand, consider the second method of pricing claims, based on a Risk-Neutral measure $q \in R^m$ based on the dual problem

$$\begin{aligned} (**)\ M^t q &= (1 + r)S_0 && (5)\\ q &> 0,\ \text{i.e., } 1 > q_j > 0. && (6) \end{aligned}$$

Given a RN measure q, the RN price of a claim $V \in R^m$ is $V_0 = \frac{1}{1+r}E_q[V]$ where r is the interest rate. Since $N' = m - r > 0$, for each RN q of the incomplete one-period market, there are uncountably many solutions $q' = q + k'$ where $k' \in Nullspace(M^t)$. Thus, for $||k'||$ small enough, q' is another RN measure. We have proved:

Lemma 3 *If an incomplete one-period market has a RN q, then $q' = q+k'$ where $k' \in Nullspace(M^t)$ is another RN measure provided $||k'||$ is small enough.*

Lemma 4 *For a incomplete one-period market with a RN q, the RN price of a claim $V \in R^m$, $V_0' = \frac{1}{1+r}E_{q'}[V]$ is well-defined if and only if $Nullspace(M^t) \perp V$.*

Proof. $V_0' = \frac{1}{1+r}E_{q'}[V] = \frac{1}{1+r}(q+k')^tV = \frac{1}{1+r}q^tV + \frac{1}{1+r}(k')^tV = V_0$ if and only if $(k')^tV = 0$ for all $k' \in Nullspace(M^t)$. ■

Since

$$Null(M^t) \oplus Range(M) = R^m \tag{7}$$

$$Null(M) \oplus Range(M^t) = R^n, \tag{8}$$

we have $Nullspace(M) \perp S_0$ if and only if $S_0 \in Range(M^t)$. And $S_0 \in Range(M^t)$ if and only if $(**)$ has a solution q (which need not be a RN measure). We have proved:

Proposition 5 *For an incomplete one-period market, the Replication price of hedgeable claim V is well-defined (agrees for all hedges X' of V) if and only if the market has a solution q to the dual problem (**).*

By $Null(M^t) \oplus Range(M) = R^m$, we have $Nullspace(M^t) \perp V$ if and only if $V \in Range(M)$. Thus,

Lemma 6 *For a incomplete one-period market with a RN q, the RN price of a claim $V \in R^m$, $V_0' = \frac{1}{1+r}E_{q'}[V]$ is well-defined if and only if V is hedgeable..*

Combining the previous proposition and lemma, and noting that

$$V_0 = X^tS_0 = \frac{1}{1+r}X^tM^tq = \frac{1}{1+r}(MX)^tq = \frac{1}{1+r}E_q[V], \tag{9}$$

we obtain the proof of the following:

Theorem 7 *In an incomplete one-period market with a RN q, every hedgeable claim V has a unique replication price $V_0 = X^tS_0$ which agrees with its RN price $V_0' = \frac{1}{1+r}E_q[V]$.*

Next, we relate these results to the concept of arbitrage-free (AF) one-period markets.

Definition 8 *An arbitrage is a hedge X for a clain $V \in R^m$ such that*
(i) $V_0 = X^tS_0 = 0$, either (ii) $V \geq 0$, and $V_j > 0$ for some $j = 1, ..., m$
or (ii)'$V \leq 0$, and $V_i < 0$ for some $i = 1, ..., m$

Definition 9 *A one-period market is AF if there are no arbitrages in it.*

Theorem 10 *If a one-period market has a RN q, then it is AF.*

Proof. By previous theorem, let $X(V)$ be an arbitrage. Then, since $q > 0$, and wlog we can take $V \geq 0$, with $V_j > 0$, we get

$$0 = X^t S_0 = \frac{1}{1+r} E_q[V] > 0 \tag{10}$$

which is a contradiction. QED. ∎

Theorem 11 *(Converse) If a one-period complete market is AF, then it has a RN q.*

Proof. Let $X(V)$ be any hedge such that $0 = X^t S_0$, then AF implies that $V = MX$ equals 0 or has mixed signed entries. That is, $U = \{V = MX \mid 0 = X^t S_0\}$ is a vector subspace of R^m that intersects the union $\overline{K} \cup -\overline{K}$ only at the origin 0, where $\overline{K}$ is the closure of the first octant. Then there is a $0 \neq q \in R^m$ such that $q \cdot V = 0$ for all $V \in U$, and $q \cdot V > 0$ (resp. < 0) if $V \in \overline{K} \setminus \{0\}$ (resp. $-\overline{K} \setminus \{0\}$) by the Separating Plane Theorem (Hahn-Banach). U separates the closed convex sets $\overline{K}$ and $-\overline{K}$. It can be normalized so that $\sum_{j=1}^{m} q_j = 1$. Also such a q must be in the interior of $\overline{K}$; otherwise if $q \in \partial \overline{K}$, then there is some $V \in \partial \overline{K}$ for which $q \cdot V = 0$. Lastly, $q^t V = q^t MX = 0$ for all V in U, implies that $q^t MX = S_0^t X$ for all $X \in W = \{X \in R^n \mid X \cdot S_0 = 0\}$ such that V is in U. This means that $M^t q \mid\mid S_0$ because $co \dim W = 1$. QED. ∎