MF Notes 2018

Part 1: Linear alg.
Part 1a One-period markets

Part 2: Probability

Part 3 & 4: Multi-period binomial
Hwk 4 due in seminar thurs week 5

Test 1 Mon week 4 closed book one hr, on hwks 1 - 3

Review thurs week 4

Office hrs week 4: tues 1 - 4 pm instead
**Part 1: Linear Algebra**

**Defn 1:** A 1-period market is \((M, \mathbf{S}_0, \mathbf{P})\) where

\[ M_{m \times n} \text{ is an } m \times n \text{ matrix of prices at } t=1 \text{ of the } n \text{ assets, } \mathbf{S}_0 \text{ is a } n \text{ column vector of } t=0 \text{ prices, and } \mathbf{P} \text{ is a } m \text{ column vector of probabilities.} \]

**Example calculation**

\[ \mathbf{M} = \begin{bmatrix} 260 & 1.01 \\ 180 & 1.01 \end{bmatrix} \]

1. Calculate \( \mathbf{S}_i \) (AAPL) = \( p^u \mathbf{S}_i^u + p^d \mathbf{S}_i^d = \frac{2}{3} \cdot 260 + \frac{1}{3} \cdot 180 \equiv \mathbf{S}_1 \)

2. Total interest rate of \( \mathbf{MM} = 1\% \) (2.8\% per annum)

**Defn 2:** A portfolio in 1-period market \((M, \mathbf{S}_0, \mathbf{P})\) is a \( n \) column vector \( \mathbf{X} = (x_1, \ldots, x_n)^t \) with \( x_j \in \mathbb{R} \) (-ve = short) representing \( x_j \) units or shares of asset \( j \).

**Example: Calculate the cost of portfolio \( \mathbf{X} = (1, 2)^t \) at } t=0 \text{ for AAPL.}

**Ans:** \( V_0 = \mathbf{X}^t \mathbf{S}_0 = \mathbf{X} \cdot \mathbf{S}_0 = \$230 + 2 \)

**Calculate the values of this portfolio at } t=1 \text{.}

\( V_1^u = \mathbf{X}^t \begin{bmatrix} 260 \\ 1.01 \end{bmatrix} = \$260 + 2 \)

\( V_1^d = \mathbf{X}^t \begin{bmatrix} 180 \\ 1.01 \end{bmatrix} = \$180 + 2 \)

**P1:** Calculate \( \mathbb{E}[V_1(\mathbf{X})] \) for AAPL with \( \mathbf{X} = (1, 2)^t \)
P2: Calculate $E[S_1]$ AAPL

P3: Calculate $\text{var}(S_1)$ AAPL

Def 3: Payoff (of portfolio $\mathbf{X}$) is an $n$-vector $\mathbf{V}_t = (V_t^1, \ldots, V_t^n)^t$ where $V_t^j$ = value of portfolio $\mathbf{X}$ in state $j$ at $t = 1$.

Example: In general market $(M_{m \times n}, \mathbf{S}_0, \mathcal{P})$, calculate $\mathbf{V}_t(\mathbf{X})$ for $\mathbf{X} = (x_1, \ldots, x_n)^t$.

Any: $\mathbf{V}_t(\mathbf{X}) = M_{m \times n} \mathbf{X} = (\sum_{j=1}^{n} x_j M_{1j}, \ldots, \sum_{j=1}^{n} x_j M_{nj})^t$

P4: Calculate $\mathbf{V}_t(\mathbf{X})$ AAPL for $\mathbf{X} = (1, 2)^t$.

Def 4: A replication/hedge of a payoff/claim $\mathbf{V}_i$ in $(M_{m \times n}, \mathbf{S}_0, \mathcal{P})$ is a portfolio $\mathbf{X}(\mathbf{V}_i)$ that satisfies: $M_{m \times n} \mathbf{X}(\mathbf{V}_i) = \mathbf{V}_i(\mathbf{X})$.

P5: Calculate the replicating/hedging portfolio $\mathbf{X}(\mathbf{V}_i)$ AAPL for $\mathbf{V}_i = (262.02, 182.02)$

P6: Calculate the same as in P5 for $\mathbf{V}_i = ($100, $200)$.

P7: Same as in P5 for $\mathbf{V}_i = ($200, $-$50).
Lectures Week 3 Fall.

**Defn**: A 1-period market is **complete** if any \( \bar{v} \in \mathbb{R}^m \) can be hedged by a portfolio \( X(\bar{v}) \) that solves \( M^X = \bar{v} \).

**Prop**: A necessary + sufficient condition for \( (M, \bar{s}_0, \bar{p}) \) to be complete is the rank(\( M \)) \( \geq m \) if \( t = 1 \) states.

**Eg**: A market \( (M_{2 \times 2}, \bar{s}_0, \bar{p}) \) where \( M_{2 \times 2} = \begin{bmatrix} S^u_1 & S^d_1 \\ S^u_2 & S^d_2 \end{bmatrix} \) is complete provided \( S^u_1 \neq S^d_2 \).

**Eg**: Consider a market \( (M_{3 \times 3}, \bar{s}_0, \bar{p}) \) with

\[
M_{3 \times 3} = \begin{bmatrix} S^u_1 & S^u_2 & 1+r \\ S^m_1 & S^m_2 & 1+r \\ S^d_1 & S^d_2 & 1+r \end{bmatrix}.
\]

It's rank(\( M \)) \( \geq 2 \).

**Suppose that** \( 2 = k \cot \frac{1}{2} \), then rank(\( M \)) = 2.

**P8**: Consider a rank 2 market \( (M_{3 \times 3}, \bar{s}_0, \bar{p}) \), calculate or characterize all \( \bar{v} \in \mathbb{R}^3 \) that can be hedged.

**P9**: In the same market as in P8, characterize geometrically the subset of \( \bar{v} \in \mathbb{R}^3 \) that cannot be hedged.
Consider $M = \begin{pmatrix} S^n & 1+2r & 1+r \\ S^m & 1+2r & 1+r \\ S^d & 1+2r & 1+r \end{pmatrix}$ for problems P10 - P14.

where $S^u > S^m > S^d > 0$; $r > 0$

Let $S_0$ be $t = 0$

price of risky asset.

P10 What is the rank of this $M_{3 \times 3}$?

P11 What are the $\mathbb{V} \in \mathbb{R}^3$ that can be hedged?

P12 Construct the payoff vector $\mathbb{V} \in \mathbb{R}^3$ corresponding to a call option on underlying asset $S_t$ with expiry $T = 1$ and strike $K = S^m$ in this market $M_{3 \times 3}$.

P13 Can the call option in P12 be hedged using $x_1$ units of the underlying asset and $x_2$ units of riskless asset with interest $r$? Calculate $x_1, x_2$.

P14 Using $M_{3 \times 3}$ and $S_0 > 0$, price the call option in P12. That is, Calculate $V_0 = x_1 S_0 + x_2$.
Replication Method for Pricing Contingency Claims (c.c.)

1. Solve \( \mathbf{x}^0 = \mathbf{V} \) for replicating portfolio \( \mathbf{x}(\mathbf{V}) \in \mathbb{R}^n \).

2. Calculate \( S_0 = \mathbf{x}^0_{\mathbf{x}(\mathbf{V})} \).

Example: To price a put on an underlying asset in a 2x2 market \( (\mathbb{H}^{2x2}, \mathbb{S}_0, \mathbb{P}) \)

with \( \mathbf{M}_{2x2} = \begin{bmatrix} \mathbb{S}_u^2 & \mathbb{S}_d^2 \\ \mathbb{S}_u^2 & \mathbb{S}_d^2 \end{bmatrix} \) \( \mathbb{S}_0 = \begin{bmatrix} \mathbb{S}_u^0 \\ \mathbb{S}_d^0 \end{bmatrix} \)

Assume \( \mathbb{S}_d^2 < \mathbb{S}_0 = \mathbb{K} < \mathbb{S}_u^2 \), and \( \mathbb{r} > 0 \), solution of (\( \mathbf{x}^0 \)) Hedging Problem yields \( \mathbb{S}_u^2 \alpha_1 + (\mathbb{r} + \mathbb{r}) \alpha_2 = \mathbb{V}_u^0 = 0 \quad (1) \)

\( \mathbb{S}_d^2 \alpha_1 + (\mathbb{r} + \mathbb{r}) \alpha_2 = \mathbb{V}_d^0 = \mathbb{K} - \mathbb{S}_d^2 > 0 \quad (2) \).

\( \alpha_1 = \frac{\mathbb{S}_d^2 - \mathbb{K}}{\mathbb{S}_u^2 - \mathbb{S}_d^2} < 0 \); \( \alpha_2 = \frac{1}{\mathbb{r} + \mathbb{r}} \left[ -\mathbb{S}_u^0 \frac{\mathbb{S}_d^2}{\mathbb{S}_u^2 - \mathbb{S}_d^2} \right] = \frac{\mathbb{S}_u^0 (\mathbb{S}_d^2 - \mathbb{S}_d^2)}{\mathbb{S}_u^2 - \mathbb{S}_d^2} > 0 \)

In other words, one needs to short \( \alpha_1 < 0 \) shares of \( \mathbb{S}_1 \), and lend/sell bonds to \( \alpha_2 > 0 \) units, thus obtaining the replication price, \( \mathbb{V}_0^{\text{put}} = \mathbb{S}_0^t (\mathbb{V}_u^0) \)

\[
\mathbb{V}_0^{\text{put}} = \frac{\mathbb{K} (\mathbb{S}_d^2 - \mathbb{K})}{\mathbb{S}_u^2 - \mathbb{S}_d^2} + \frac{\mathbb{S}_u^0 (\mathbb{S}_d^2 - \mathbb{K})}{\mathbb{r} + \mathbb{r}} \left( \frac{\mathbb{S}_u^2 - \mathbb{S}_d^2}{\mathbb{S}_u^2 - \mathbb{S}_d^2} \right) = \frac{(\mathbb{K} - \mathbb{S}_d^2) (\mathbb{S}_u^0 - \mathbb{K})}{\mathbb{r} + \mathbb{r}} \left( \frac{\mathbb{S}_u^2 - \mathbb{S}_d^2}{\mathbb{S}_u^2 - \mathbb{S}_d^2} \right)
\]

From \( \mathbb{V}_0^{\text{put}} \) we derive a necessary condition for the replication price of a put option on \( \mathbb{S}_1 \) with strike \( \mathbb{K} = \mathbb{S}_0 \) to make financial sense: \( \mathbb{S}_u^0 > (\mathbb{r} + \mathbb{r}) \mathbb{S}_0 \).
This is one half of the no-arbitrage condition on a 2x2 market.

By doing a similar calculation for a call with strike $K = S_0$ in the same 2x2 market, we derive the other half: $S_{1d} < S_0(1+r)$.

P15: Compute the replication price of a call at strike $K=S_0$ in the above 2x2 market.

P16: Derive the condition $S_{1d} < S_0(1+r)$ in order for $V_0^{\text{call}}$ in P15 to make financial sense.
In a 1-period market \((M_{\text{maxn}}, S_0, \mathbb{P})\), a r.v. \(X_t\) at \(t=1\) (e.g. \(S_t, V_t, \eta_t\), etc) is called a \(\mathbb{P}\)-Martingale if \(E^\mathbb{P}[X_t] = \sum_{j=0}^{\infty} p_j X(t) = X_0\), its value at \(t=1\)

**Eg 1:** Let \(S_0 = 1\), \(S_t = (\frac{2}{3})\), \(\mathbb{P} = (\frac{2}{3}, \frac{1}{3})\) \(\Rightarrow S_t\) is \(\mathbb{P}\)-Martingale.

**Eg 2:** Let \(S_0 = 230\), \(S_t = [260]\), \(\mathbb{P} = (\frac{2}{3}, \frac{1}{3})\) \(\Rightarrow S_t\) \(\not\in\) \(\mathbb{P}\)-Martingale.

**P1:** Calculate \(\mathbb{P} = (P_1, P_2)\) such that \(P_j > 0\), and \(P_1 + P_2 = 1\), for which \(S_t = [260]\), \(S_0 = 230\) is a \(\mathbb{P}\)-Martingale.
The Dual Problem for \((M_{mxn}, \overline{S}_0)\)

\[(**) \quad \overline{A}^t \overline{M}_{mxn} = (1+r) \overline{S}_0^t \quad \overline{A}^t = (a_{ij}, \ldots, a_{Em})\]

Equivalent to:

\[(**') \quad M_{mxn}^t \overline{A} = (1+r) \overline{S}_0\]

which is in turn a simultaneous eqns in \(m\) unknowns \(\overline{A}\):

They are:

1. \(E_{\overline{A}}[S_i] = (1+r)\overline{S}_0\)
2. \(E_{\overline{A}}[S_k] = (1+r)\overline{S}_0\)
3. \(\sum \overline{a}_{ij} = 1\)
4. \(S_k(\overline{A})\) with price vector \(S_k^t = (S_{k1}, \ldots, S_{km})\)

Solvability of \((**')\) for a complete 1-period market:

Since \(\text{col rank}(M_{mxn}) = \text{row rank}(M_{mxn}) = \text{col rank}(M_{mxn}^t) = m\) by completeness,

if \(n \leq m\), \((**')\) is solvable for \(\overline{A}\).

Define a Risk-Neutral Prob. measure is a \(\overline{A} \in \mathbb{R}^m\) that satisfies \((**')\)

\[\text{and } \overline{a}_{ij} > 0 \quad \text{for } j = 1, \ldots, m\]

Geometrically, one requires \(\overline{A}\) to be in the interior (1st Octant) in \(\mathbb{R}^m\).
For $2 \times 2$ complete market

$$M_{2x2} = \begin{bmatrix} S^u & 1+r \\ S^d & 1+r \end{bmatrix}, \quad \mathcal{F}_0 = \begin{pmatrix} \mathcal{F} \\ 0 \end{pmatrix}$$

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Yields:

$$q_1 = \frac{(1+r)S_0 - S^q}{S^u - S^d}$$

$$q_2 = \frac{S^u - S_0(1+r)}{S^u - S^d}$$

Note that in order for $\mathcal{F}^\mathcal{C} = (q_1, q_2)^T$ to be in int $(1st \ 2nd)$, n.a.s.c. are

(i) $S^d < (1+r)S_0$, (ii) $(1+r)S_0 < S^u$.

These algebraic/financial conditions will reappear in the Theory for No-Arbitrage in $2$-period markets.

19. Calculate a Risk-Neutral Probability $\mathcal{F}^\mathcal{C} = (q_1, q_2)^T$ with $q_1 > 0$, $q_2 > 0$ and $q_1 + q_2 = 1$

for the market AAPL

19. Calculate $V_0 = \mathbb{E}_{\mathcal{F}}[\mathcal{V}]$ where

$\mathcal{V}$ is the Call on AAPL with strike $K = S_0 = 230$. 

\[ \mathbb{E} \]
Next we consider the Risk-Neutral Problem geometrically, first for the complete 2×2 market $\begin{bmatrix} S_0^u & 1+r \\ S_0^d & 1+r \end{bmatrix}$.

$M$ acts on $x \in \text{int}(1st\ quad)$ and sends it to $\tilde{x}_0 = \begin{pmatrix} S_0^u > 0 \\ 1 \end{pmatrix}$, also in $\text{int}(1st\ quad)$.

Note that for a complete market, $\det(M_{2\times2}) = \det(M_{2\times2}) = (S_1^u - S_1^d)(1+r) > 0$.

In other words, $M_{2\times2}$ and $M_{2\times2}^t$ are orientation-preserving transformations.

For the 2×2 market that is complete, the characteristic polynomial is

$0 = \det(M_{2\times2} - \lambda I) = \det\begin{bmatrix} S_0^u - \lambda & 1+r \\ S_0^d & 1+r - \lambda \end{bmatrix}$

$= (S_0^u - \lambda)(1+r - \lambda) - S_0^d(1+r)$

$= \lambda^2 - \lambda (S_0^u + 1+r) + (S_0^u - S_0^d)(1+r)$

$= \lambda^2 - \lambda (\text{tr}M_{2\times2}) + \det M_{2\times2}$

$\lambda = \frac{\text{tr}M_{2\times2} \pm \sqrt{\text{tr}^2(M_{2\times2}) - 4\det M_{2\times2}}}{2}$

Thus, the eigenvalues are real if $\text{Discriminant} = \text{tr}^2(M_{2\times2}) - 4\det M_{2\times2} > 0$. Under these conditions, there is a similarity transformation.

$M_{2\times2} = M_{2\times2} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

General question: What is the financial role if any of $\lambda_1, \lambda_2 \in \mathbb{R}$?
Besides the hedging prob. (* \( M_{\frac{\delta}{\gamma}} = \hat{\gamma} \)) used to calculate a hedging portfolio \( \hat{\gamma} \) for payoff \( \hat{\gamma} = \mathbf{1}^T \), there is another significant linear system equation (**) that is important in finance:

\[
\mathbf{Q}^T \mathbf{M} = (1+r) \mathbf{S}_0 \tag{**}
\]

where \( \mathbf{Q}^T = (Q_1, \ldots, Q_m) \in \mathbb{R}^m \) is a left-solution to the matrix of the market.

More constraints on \( \mathbf{Q} \) will be needed:

1. To make \( \mathbf{Q} \) a probability vector, \( Q_j \geq 0 \) and \( \sum Q_j = 1 \)

2. To make \( \mathbf{Q} \) a risk-neutral probability vector, one needs \( Q_j > 0 \) and \( \sum Q_j = 1 \)

Clearly (**) is equivalent to \( \mathbf{M}^T \mathbf{Q} = (1+r) \mathbf{S}_0 \tag{**}' \)

It is therefore a system of \( n \) eqns. for \( m \) variables \( Q_j \).

Eq 1: Consider market \( \left( \mathbf{M}_{2 \times 2}, \mathbf{S}_0, \mathbf{P} \right) \) where \( \mathbf{M}_{2 \times 2} = \begin{bmatrix} S_{11} & 1+\gamma_1 \\ S_{12} & 1+\gamma_2 \end{bmatrix} \)

Then \( \mathbf{M}^T \mathbf{Q} = (1+r) \mathbf{S}_0 \tag{**}' \)

Then \( \left( 1, S_{11}, S_{12} \right) \) becomes the simultaneous eqns.

\( \begin{align*}
(1) & \quad q_1, S_{11} + q_2, S_{12} = (1+r)S_0 \\
(2) & \quad (1+r)(q_1 + q_2) = (1+r)
\end{align*} \)

Note \((1) \iff q_1 + q_2 = 1 \tag{2}'\)
\( q_2 = 1 - q_1 \) \hspace{.5cm} \text{Substitute in (1)}

\[
q_1 S_i^u + (1-q_1) S_i^d = (1+r) S_0 \quad (1) \Rightarrow q_1 = \frac{(1+r) S_0 - S_i^d}{S_i^u - S_i^d}
\]

\[
q_2 = \frac{S_i^u - (1+r) S_0}{S_i^u - S_i^d}
\]

Assuming \( S_i^u > S_i^d \) for completeness \( q (M \in \mathbb{R}^2, \{ S_0, P \}) \)

We note that \( \text{n.a.c.} \) \hspace{.5cm} \text{(necessary and sufficient conditions)}

for \( q_1 > 0 \) and \( q_2 > 0 \) are the same as previously found in the 2x2 market for financial sense of the replication prices of calls, and put on \( S_i \), with strike \( K = S_0 \), namely:

(a) \( S_i^u > (1+r) S_0 \) \hspace{1cm} (b) \( (1+r) S_0 > S_i^d \)

In general 1-period market \( (M^{mxn}, \overrightarrow{S_0}, \overrightarrow{P}) \)

we have (\#\#):

\[
\overrightarrow{Q} \overrightarrow{M} = (1+r) \overrightarrow{S}_0 \quad \text{or}
\]

\[
\overrightarrow{Q} \overrightarrow{M} = (1+r) \overrightarrow{S}_0 \quad \text{which is equivalent to}
\]

system of simultaneous \( \overrightarrow{Q} \)s:

1. \( \sum_{j=1}^{n} q_j \overrightarrow{M}_{j1} = (1+r) \overrightarrow{S}_0^{(1)} = \overrightarrow{Q} \cdot \overrightarrow{S}_1 = \overrightarrow{Q} \cdot \overrightarrow{M}_1 
\]
2. \( \sum_{j=1}^{n} q_j \overrightarrow{M}_{j1} = (1+r) \overrightarrow{S}_0^{(2)} \quad \text{where} \overrightarrow{S}_0^{(k)} \text{is the } k^{\text{th}} \text{term in} \overrightarrow{S}_0 \).

3. \( \overrightarrow{Q} \cdot \overrightarrow{S}_k = \overrightarrow{Q} \cdot \overrightarrow{M}_k 
\]

4. \( \sum_{j=1}^{n} q_j = 1 \)

Here \( \overrightarrow{M}_k \) is \( k^{\text{th}} \) col. \( \overrightarrow{Q} \overrightarrow{M}_{mxn} = \overrightarrow{S}_k \) price vector at \( t=1 \)

\[
S_0 \overrightarrow{M}_n = (1+r, \ldots, 1+r)^T \text{ is } t=1 \text{ price of bond}
\]
Note that the scalar (inner or dot) product \( \mathbf{a} \cdot \mathbf{s}_k \)

between the \( m \)-vectors \( \mathbf{a} \) and \( t=1 \) price vector \( \mathbf{s}_k \) of both asset

\[ \mathbf{a} \cdot \mathbf{s}_k = \sum_{j=1}^{m} a_j s_j(1) = E_{\mathbf{a}}[s_k] \]

when by abuse of notation the price vector \( \mathbf{s}_k \) is identified with the random variable \( s_k \), the price at \( t=1 \) of asset \( k \).

Thus, we have shown that (**) is equivalent to:

\[ \frac{1}{1+r} E_{\mathbf{a}}[s_k] \triangleq E_{\mathbf{a}}[\mathbf{s}_k] = s_k(0), \text{ for } t=0 \] (pure state) asset

where \( \mathbf{s}_k = s_k(0) \).

In other words, a solution to (**) for \( \mathbf{a} \in \mathbb{R}^m \) is a probability vector that can be a possible candidate for the (already defined) Martingale/Risk Neutral Probability measure, that simultaneously satisfy (n-1) condition for the Martingale Property of all (n-1) risky asset.
Consider Solvability of (**) with fixed vector \((\mathbf{f}_0) = \mathbf{f}_0\). Since col. rank \((M_{m \times n}) = \text{row} - \text{rank} (M_{m \times n})\), we deduce that \(M_{2 \times 2}^T\) has 2 linearly independent columns as long as \(s_i > s_i^d\). Thus there is always a solution \(\mathbf{G} \in \mathbb{R}^2\) such that \(q_1 + q_2 = 1\), \(\mathbf{G}^T \mathbf{f} = (1+t)\mathbf{f}_0\).

For all \(t = 0\) pure vector \(\mathbf{f}_0 = (\mathbf{f}_0)\).

Note the important point that any solution to (**) for \(\mathbf{G} \in \mathbb{R}^m\) does not mean \(q_j > 0\) for \(j = 1, \ldots, m\). In fact, not even \(q_j > 0\) for all.

Only vectors with \(\mathbf{G} > 0\), i.e., \(q_j > 0\) for all \(j = 1, \ldots, m\) are Risk-Neutral measures. This is captured geometrically by the requirement that \(\mathbf{G}\) is in the interior of the 1st Octant (it does not belong to the boundary faces nor axes of 1st Octant).
Given that \((M = \mathbb{R}^{1 \times 1} \mathbb{S}_0)\) has a Risk Neutral \(\mathbb{Q}\), there are several consequences which follow:

(i) **Thm:** Every \(V \in \mathbb{R}^2\) after discounting by \(1+r\) is a \(\mathbb{Q}\)-Martingale.

**Pf:** Existence of RN \(\mathcal{Q}\) \(\Rightarrow \mathbb{S}_0(1+r) < \mathbb{S}_u\) and \(\mathbb{S}_d < \mathbb{S}_0\) \((1+r)\).

which in turn \(\Rightarrow \mathbb{S}_u > \mathbb{S}_d\). Thus, by completeness of \((\mathbb{R}^{1 \times 2} \mathbb{S}_0)\), every \(V \in \mathbb{R}\) has a hedge \(\mathbb{X}(V)\)

\[ S + M \mathbb{X} = V \] and a replication price $V_0 = \mathbb{S}_0 \mathbb{X}(V)$. We now show that

\[ E_Q[V] = \mathbb{Q}^t \mathbb{V} = \mathbb{Q}^t M \mathbb{X}(V) = (1+r) \mathbb{S}_0 \mathbb{X}(V) \]

\[ \Rightarrow E_Q[V] = \mathbb{Q}^t (\frac{V}{1+r}) = V_0 \quad QED. \]

(ii) **Cor** In \((\mathbb{M} = \mathbb{R}^{1 \times 2} \mathbb{S}_0)\) with RN \(\mathcal{Q}\), the replication price \(SV_0 = E_Q[\frac{V}{1+r}]\) for each \(\mathbb{V} \in \mathbb{R}^2\).
Let 

\[ M = \begin{pmatrix} 260 & 1.0 \\ 180 & 1.01 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 219 \\ 1 \end{pmatrix} \]

Price can at strike \( K = 220 \).
No Arbitrage Principle

An arbitrage in $(M, x, S_0, \mathbb{P})$ is a portfolio $\mathbf{X}$ s.t.

(i) $X_0 = S_0^x \mathbf{X} = 0$

(ii) $0 \leq \sum_{i=1}^{m} V_i = \frac{m}{\mathbb{E}} \mathbf{X}$

(iii) $X_j > 0$ for at least one $j = 1, \ldots, m$

Eq. In $2 \times 2$ case, $(M, x, S_0)$ complete, an arbitrage is a portfolio $\mathbf{X} = (X_1, X_2)$ s.t. $X_1 S_0 + X_2 = 0$ (1)

Equation 2

$\mathbb{V} > 0$ for $V_1 > 0$ or $V_2 > 0$.

Def.: Market $(M, S_0)$ is arbitrage-free or AF if there are no arbitrages in it.

P21: Calculate the RN price of a call at strike $K$ s.t.

$S_d < K < S_u$ in a $2 \times 2$

market $(M_{2 \times 2}, S_0)$, for which there is a $\mathbb{Q} > 0$ satisfying $\mathbb{Q} = \left(1 + r \right) S_0^x$

P22: Do P21 but for a put at same $K$. 
Thm: n.a.s.c. for \( (\mathbb{L} \times \mathbb{L}, S_0) \) to be AF is

(i) \( S_0(1+r) < S_i \)

(ii) \( S_i < S_0(1+r) \)

PF: By defn of an arbitrage \( X \in \mathbb{R}^2 \) we have

\[
S_0 X = S_0 x_1 + x_2 = 0 \implies x_2 = -S_0 x_1.
\]

Substituting in

\[
\nabla(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

we get

\[
v_i^u = S_i x_1 + (1+r)(-S_0 x_1) = x_1 (S_i^u - (1+r) S_0) \geq 0 \quad (1)
\]

\[
v_i^d = S_i^d x_1 + (1+r)(-S_0 x_1) = x_1 (S_i^d - (1+r) S_0) \leq 0 \quad (2)
\]

if \( \text{n.a.s.c. (i)+(ii) holds.} \)

Next note that (i)+(ii)

\[
\Rightarrow X \notin \text{Oct} \Rightarrow = \text{closure of 1st quad.}
\]

thus not an arbitrage
p23: Construct an arbitrage
\[ \hat{x} = (x_1, x_2) \] when
\[ A = S_i^u - S_o(1+r) > 0 \]
\[ B = S_i^u - S_o(1+r) > 0 \]
Give answer for \( x_1, x_2 \)
in terms of \( A, B \)

p24: Give conditions
in terms of \( A, B \) in p23
in order for \((M_{2x2}, \hat{x})\) in p23
to be complete
\[
\begin{align*}
N(M) = \{ x \mid M x = \bar{b} \} \text{ is a hyperplane of } K^n.
\end{align*}
\]

Suppose \( \dim X \).

\[
\text{Re}(M) = \left\{ x \in K^n \mid \exists x \in K^n, \bar{z} = M x \right\} \subseteq K^n.
\]

**Theorem:** \( N(M) \cap \text{Re}(M) = K^n \)

\[
\dim \text{Re}(M) = n-k = m-k' = r
\]

\[
\dim \text{Re}(M) = \text{rank}(M) = \text{rank}(M^t) = r
\]

**Applied:**

\[
\begin{align*}
\text{Suppose } & K = N(M) \text{ has } \dim K = k. \\
\text{Then, if } & \bar{v} \in \text{Re}(M), \bar{v} \in \text{Comp}(M)
\end{align*}
\]

and \( x(\bar{v}) \) is a solution, then \( \bar{x}' = \bar{x} + \bar{n}, \bar{n} \in K \\
also solves \( M \), i.e., \( \bar{M} \bar{x}' = \bar{M} \bar{x} + \bar{M} \bar{n} = \bar{v} + \bar{n} \).

Financial problems potentially arise. In case \( N(M) \) nontrivial \( (\dim = k) \) \( (\text{Re}(M^t)) \) has \( \dim = n-k \).

**Lemma:**

\[
\begin{align*}
&\text{By the replication method, } \forall x(\bar{v}) = X^t S_0 = V_1 = (x^t S_0) = X^t S_0 + n^t S_0
\end{align*}
\]

\[
\Rightarrow \text{recovery condition, } \forall x(\bar{v}) \text{ be } \bar{v} \text{ is } S_0 \perp K.
\]

\[
(\forall x) \text{ } M^t \bar{v} = (H H) S_0.
\]

**Multiplication:** \( M^t S_0^t N(M^t) = K' \) has \( \dim = k = m - n + k > 0 \) \( (n \geq m, k \leq n - m) \)

If \( \bar{v} \) solve \( G^t \), then \( \bar{v} = \bar{v} + \bar{n} \) \( \text{such that } \bar{n} \in K' = N(M^t) \)

\[
\text{solve } \bar{v} \text{ and } M^t \bar{v} = n_0 + n \bar{n}' = (H H) S_0
\]

**Prop:**

If \( \text{rank}(M) < m \) (completeness \( m - r = l > 0 \)), then \( k = m - r = l > 0 \\
then \( \dim K(M^t) = \dim K' = l > 0 \) it is replication method have multiple \( \bar{v} \) if there is one!

**Prop:**

\[
\begin{align*}
&\text{Suppose } \bar{x} \text{ has a } \bar{v} \text{ (} \text{Re}(M^t)) \text{, then if } k' = m-r = l > 0 \text{ (may be)}
\end{align*}
\]

and \( m \geq \text{ incomplete, then } \text{ (a) } K = N(M) \) has \( \dim k = n-r > l > 0 \\
\text{(b) Replicate process are!}
\]

**Proof:**

\[
\begin{align*}
&\text{Suppose } \bar{x} \text{ has a } \bar{v} \text{ (} \text{Re}(M^t)) \text{, then if } k' = m-r = l > 0 \text{ (may be)}
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\]

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and \( m \geq \text{ incomplete, then } \text{ (a) } K = N(M) \) has \( \dim k = n-r > l > 0 \\
\text{(b) Replicate process are!}
\]

Then: If \( n_r > m_r > 0 \), and \( \exists \bar{q} \) sol of (x^s) that is RN, new \( \{ \bar{q}^s \} \) only need \\
the replicat. prin. = RN prin.

**Pf:** \( SV_0 = X^r S^r_0 = \bar{X}^r \bar{m}^r \bar{q}^r / (1+r) = (M^r \bar{m}^r \bar{q}^r / (1+r) = \bar{v}^r \bar{q}^r / (1+r) \)

**Cor:** \( \frac{1}{1+r} \frac{1}{r} E_{v^r} [v^r] \)

By prop, any replicat. prin.

\( = (x^s+\bar{m}^s) S^s_0 = \forall \text{ all } RN \) prin.

**Pf:** Since \( k^s = m_r - n_r > 0 \) and \( k = n-r > 0 \), and \( \forall (\bar{q}^s) \in w^r \) \( \forall (w) \) for behav. \( v^r \in q^r \), and that \( v^r \) must. \( \bar{q}^r = \bar{v}^r + \bar{m}^r \) sol (to (v)) since one of \( q^r \) is RN.

For supp. small \( \| q^r \| \), \( \bar{q}^r \) is RN, so

\( \bar{v}^r + \bar{m}^r \) IQ \( \bar{v}^r = \frac{\bar{v}^r + \bar{m}^r \bar{q}^r / (1+r)} {1+r \frac{1}{r} E_{v^r} [v^r]} \)

\( = E_{v^r} [v^r] + 0 \) since \( (\bar{m}^r)^t \bar{v}^r = (\bar{m}^r)^t \bar{X}^r = (M^r \bar{m}^r \bar{q}^r)^t = (\bar{q}^r)^t = 0. \)