Energy-enstrophy theory for coupled fluid / rotating sphere system - exact solutions for super-rotations

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1 Introduction

This paper offers a short review [14] of some recent advances as well as new results [5], [7], [6], [15], [21], [22], [9], [8] in the application of equilibrium statistical mechanics to complex geophysical and astrophysical flows [1], [13] including but not restricted to the super-rotation of the Venusian middle atmosphere [3]. An important overall aim of our results is the extension of statistical equilibrium theories to macroscopic flows that lack some of the basic properties previously assumed to be necessary such as a conserved Hamiltonian and selected invariants - global angular momentum and enstrophy. Specific problems to which we address this review include global scale flows coupled to a rotating solid sphere by complex torque mechanisms.

Decades of research show that the standard statistical equilibrium approach have produced some scientific progress in the simplest cases such as flows in a periodic square, in the unbounded plane and uncoupled flows on a sphere [11], [25], [4], [10], [1]. This approach is somewhat more successful in predicting the large-scale coherent end products of relaxing / decaying 2d turbulence - these are features which are independent of initial conditions - than details of their inertial range spectra and fine-scale structures. In particular, the location and orientation of kinks in the numerically simulated spectra of simple 2d turbulent flows appear to be sensitive to the initial energy distributions [2].

We will discuss other reasons for current dis-satisfaction with this classical approach and introduce much needed extensions to the standard statistical equilibrium approach. These modifications yield exactly-solvable theories which are nontrivial (meaning non-Gaussian and not necessarily mean field) and which appear to predict correctly some of the transition boundaries between super and sub-rotational end-states in coupled fluid-sphere systems (cf. main theorem parts C and D). In view of their complete integrability, it should be possible to work out the details of the energy spectra of these models.

One of the main discovery of this new approach is not only that when energy is allowed to flow in and out of the spherical harmonics (with total wavenumber $l = 1$) that carry angular momentum, energy piles up in the spherical harmonics with lowest wavenumber $l = 1$ allowed by the condition of zero total circulation, but that there is an interesting asymmetry between the pro-rotating and counter-rotating flow states. Indeed, we find (and prove) the first ever instance of a phase transition at a positive critical temperature in a statistical equilibrium theory of 2d macroscopic flows - the transition is between disordered flow states and the counter-rotating state. Whether this statistical equilibrium approach is valid for certain specific regimes within the non-equilibrium phenomena of 2d turbulence will depend on future detailed comparisons between its predictions and observations of planetary atmospheres and the outcomes of sophisticated DNS studies of 2D Navier Stokes systems in complex domains (cf. Ditlevsen [23] for a discussion of the conditions for equilibrium versus energy-enstrophy cascades in the shell models [24]).

Fjortoft's and later Kraichnan's identification of energy inverse cascades in nearly inviscid quasi-2D turbulence - a non-equilibrium result - renewed interest in Onsager's approach [11] using equilibrium statistical mechanics [25], [20]. Some of these works are based on the Lagrangian vortex gas methodology [11], [25], [20]. Other works are based mainly on the spectral and Eulerian forms of the classical energy-enstrophy theory of Kraichnan [4], [1]. The vortex gas models with fixed numbers of particles, impose constraints on energy and angular momentum in the mean.

Most of the classical work including [25] is based on a microcanonical ensemble for the energy of the system - it was thought that an energy resevoir could not have negative temperatures even if it is possible to locate such an energy bath in unbounded flows. Clearly for coupled systems with complex boundaries - where in fact the energy and angular momentum of the fluid component is of greater importance than the combined energy/ angular momentum of the coupled system - it is natural to use a canonical-in-energy Gibbs ensemble. The earlier doubts about negative- temperature heat baths can be dispelled by thinking of temperature as the measure of mean kinetic energy, and negative temperatures as measuring very high kinetic energies - that is, energies above the threshold level beyond which the entropy of the system decreases as energy increases.

It is thus natural to think of the fluid component of the coupled system as being in contact with an energy / angular momentum bath situated in the infintely massive sphere. The total circulation of the fluid in the coupled system is a conserved quantity - it is zero by Stokes theorem. The successful justification for applying classical statistical equilibrium theories to coupled macroscopic flows - one that is based on the existence of two widely separated time scales - can also be used to justify keeping the enstrophy (but not the higher moments [19], [17]) fixed in the eddy relaxation time scale. Because the doubly canonical form of its partition function makes the classical energy-enstrophy theories [4]

Gaussian and therefore not well defined at low temperatures [5], there is actually only one viable option left for a energy-enstrophy theory of coupled fluid system with complex boundaries, namely one that is canonical in energy and microcanonical in enstrophy.

In summary we offer (A) a formulation of correct and solvable statistical mechanics theories of geophysical flows based on energy, enstrophy, total circulation and non-conservation of angular momentum and (B) exact closed form solutions of these models [15]. The specific implementation of microcanonical enstrophy constraints in this approach leads to (B) - exact solutions of the resulting theories using the Kac-Berlin method [12] of steepest descent for spherical models. The main point discussed below in further detail is that these spherical models fix the low temperature problems of the classical energy-enstrophy theories [4] and yet are solvable in closed form. Using an energy functional that is not Hamiltonian does not present problems for the statistical mechanics approach in general - it requires only a partition function based on an action and constraints that are defined in overall phase space. We will derive in this paper such a generalized energy functional for a coupled geophysical flow that is not a Hamiltonian [18].

2 Coupled Barotropic Fluid - Rotating Sphere Model.

Consider the system consisting of a rotating massive rigid sphere of radius R, enveloped by a thin shell of non-divergent barotropic fluid. The barotropic flow is assumed to be inviscid, apart from an ability to exchange angular momentum and kinetic energy with the infinitely massive solid sphere through a complex torque mechanism. We also assume that the fluid is in radiation balance and there is no net energy gain or loss from insolation. This provides a crude model of the complex planet - atmosphere interactions, including the enigmatic torque mechanism responsible for the phenomenon of atmospheric super-rotation - one of the main applications motivating this work.

For a geophysical flow problem concerning super-rotation on a spherical surface, one of the key parameters is angular momentum of the fluid. In principle, the total angular momentum of the fluid and solid sphere is a conserved quantity but by taking the sphere to have infinite mass, the active part of the model is just the fluid which relaxes by exchanging angular momentum with an infinite reservoir. The rest frame energy of the fluid and sphere is conserved. Again we need only keep track of the kinetic energy of the barotropic fluid - in the non-divergent case, there is no gravitational potential energy in the fluid because it has uniform thickness and density, and its upper surface is a rigid lid.

The rest frame kinetic energy of the fluid expressed in a frame that is rotating at the angular velocity of the solid sphere is

$H_T[q] = \frac{1}{2}\int_{S^2} dx \left[(u_r + u_p)^2 + v_r^2\right] = -\frac{1}{2}\int_{S^2} dx\ \psi q + \frac{1}{2}\int_{S^2} dx\ u_p^2$

where u_r, v_r are the zonal and meridional components of the relative velocity,

u_p is the zonal component of the planetary velocity (the meridional component being zero since planetary vorticity is zonal), and ψ is the stream function for the relative velocity. It is convenient to work with the pseudo-energy as the energy functional for the model,

$H[w] = -\frac{1}{2}\int_{S^2} dx\ \psi q = -\frac{1}{2}\int dx\ \psi(x)w(x) - \Omega \int dx\ \psi(x)\cos\theta.$

Relative vorticity circulation in the model is fixed to be $\int w dx = 0$. The second term in the energy is equal to 4Ω times the variable angular momentum density of the relative fluid motion and has units of m^4/s. The only mode in the eigenfunction expansion of w that contributes to its net angular momentum is $\alpha_{10}\psi_{10}$ where $\psi_{10} = a\cos\theta$ is the first nontrivial spherical harmonic; it has the form of solid-body rotation vorticity.

3 Heisenberg-Ising Model

Given N fixed mesh points x_k on S^2 and the Voronoi cells based on this mesh [9], we approximate the relative vorticity by discretizing the scalar vorticity field as a piecewise constant function, $\omega(x) = \sum_{j=1}^{N} s_j H_j(x)$, where $s_j = \omega(x_j)$ and $H_j(x)$ is the characteristic function for the domain D_j, that is

$$H_j(x) = \begin{cases} 1 & x \in D_j \\ 0 & otherwise \end{cases} .$$

Vectorizing in a trivial way leads to a useful Heisenberg-Ising model- represent the site vorticity by the vector $\vec{s}_j = s_j\vec{n}_j$ where $\vec{n}_j$ denotes the outward unit normal to the sphere S^2 at x_j; represent the spin Ω of the rotating frame by the vector $\vec{h} = \frac{2\pi}{N}\Omega\vec{n}$ where $\vec{n}$ is the outward unit normal at the north pole of S^2; and denote by γ_{jk} the angle subtended at the center of S^2 by the lattice sites x_j and x_k, to obtain the following Heisenberg-Ising spin-lattice model for the total (fixed frame) kinetic energy of a barotropic flow in terms of a rotating frame at spin rate Ω,

$$H_H^N = -\frac{1}{2}\sum_{j\neq k}^{N} J_{jk}\vec{s}_j\cdot\vec{s}_k + \vec{h}\cdot\sum_{j=1}^{N}\vec{s}_j \qquad (1)$$

where the interaction matrix is now given by $J_{jk} = \frac{16\pi^2}{N^2}\frac{\ln(1-\cos\gamma_{jk})}{\cos\gamma_{jk}}$, and the dot denotes the inner product in R^3.

The Kac-Berlin method [12] is modified [22] to solve the spherical Heisenberg-Ising model which consists of H_H^N, the spherical or relative enstrophy constraint, $\frac{4\pi}{N}\sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j = Q$ and $\frac{4\pi}{N}\sum_{j=1}^{N}\vec{s}_j\cdot\vec{n}_j = 0$. Looking ahead, we note the important fact that the following vectorial sum or magnetization $\Gamma = \frac{4\pi}{N}\sum_{j=1}^{N}\vec{s}_j$ will turn out to be a natural order parameter for the statistics of barotropic flows coupled to a massive rotating sphere.

4 Solution of the spherical model

This family of spherical Heisenberg-Ising models for barotropic vortex statistics allows us to model the thermal interactions between local relative vorticity $\omega(x)$ and a kinetic energy reservoir at any fixed temperature T. The spherical constraint enforces the microcanonically fixed relative enstrophy $Q > 0$ but allows angular momentum in each of the three principal directions to change. Similar to the equilibrium condensation process found in the case $\Omega = 0$ for the spherical Ising model [21], [22] kinetic energy of barotropic flow settles into a Goldstone symmetry-breaking ground state at numerically-very-small, negative temperatures $T_c < T < 0$ (associated with extremely large energies). Unlike the $\Omega = 0$, there is no 3-fold degeneracy in the Goldstone modes and only the mode ψ_{10} which carries angular momentum that is aligned with the rotation axis $\Omega\vec{n}$, has a large amplitude.

The exact solution of the spherical Heisenberg-Ising models H_H^N proceeds along similar lines to the Kac-Berlin method for the spherical Ising model. In the thermodynamic or continuum limit as $N \to \infty$, the partition function is calculated using Laplace's integral form,

$$\begin{aligned} Z_H^N &\propto \int D(\vec{s}) \exp\left(-\beta H_H^N(\vec{s})\right) \delta\left(Q\frac{N}{4\pi} - \sum_{j=1}^{N} \vec{s}_j \cdot \vec{s}_j\right) \\ &= \int D(\vec{s}) \exp\left(\frac{\beta}{2}\sum_{j\neq k}^{N} J_{jk}\vec{s}_j \cdot \vec{s}_k - \beta\vec{h}\cdot\sum_{j=1}^{N}\vec{s}_j\right) \times \\ &\left(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta\left(N - \frac{4\pi}{Q}\sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j\right)\right)\right) \end{aligned}$$

Thus, the partition function has the form

$$\int D(\vec{s}) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left(N\left(\eta - \frac{1}{N}\sum_{j\neq k}^{N} K_{jk}(Q,\beta,\eta)\ \vec{s}_j\cdot\vec{s}_k - \frac{\beta}{N}\vec{h}\cdot\sum_{j=1}^{N}\vec{s}_j\right)\right)$$

where

$$K_{jk}(Q,\beta,\eta) = \left\{ \begin{array}{cc} \frac{4\pi}{Q}\eta & j=k \\ -\frac{\beta}{2}J_{jk} & j\neq k \end{array} \right\}.$$

Solution of the Gaussian integrals requires diagonalizing the interaction in H_H^N in terms of the spherical harmonics $\{\psi_{lm}\}_{l=1,}^{\infty}$ which are natural Fourier modes for Laplacian eigenvalue problems on S^2 with zero circulation, that is $\vec{s}_j = \vec{n}_j \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \alpha_{lm}\psi_{lm}(x_j)$ where α_{lm} are the Fourier amplitudes, $-\frac{1}{2}\sum_{j\neq k}^{N} J_{jk}\ \vec{s}_j\cdot\vec{s}_k = \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\lambda_{lm}\alpha_{lm}^2$ and $\vec{h}\cdot\sum_{j=1}^{N}\vec{s}_j = \frac{1}{2}\Omega C\alpha_{10}$ where the eigenvalues of the Green's function for the Laplace-Beltrami operator on S^2 are $\lambda_{lm} = \frac{1}{l(l+1)}$, $l = 1, ..., \sqrt{N}$, $m = -l, ..., 0, ..., l$. Thus,

$\frac{1}{N}\sum_{j\neq k}^{N} K_{jk}(Q,\beta,\eta)\ \vec{s}_j\cdot\vec{s}_k = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm} + \frac{\eta}{Q}\right)\alpha_{lm}^2.$

4.1 Restricted partition function and non-ergodic modes

Next we write the problem in terms of the restricted partition function

$$\int \prod_{m=-1}^{1} d\alpha_{1m} \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \; \exp\left\{ N \left[\begin{array}{c} \eta - \frac{\beta}{2N}\Omega C \alpha_{10} \\ -\left(\frac{\beta}{4N} + \frac{\eta}{Q}\right) \sum_{m=-1}^{1} \alpha_{1m}^2 \end{array} \right] \right\}$$

$$\int D_{l\geq 2}(\alpha) \exp\left(-N \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left(\frac{\beta}{2N}\lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2 \right).$$

Because of non-ergodicity of the condensed modes, we should not integrate over the ordered modes in this problem, namely α_{1m}, which are the amplitudes of the 3-fold degenerate ground modes ψ_{1m} that carry global angular momentum. We show in [15] that only one single class of modes can have nonzero amplitudes in the condensed phase of this problem, namely those belonging to the meridional wave number $l = 1$. The statistics of the problem are therefore completely determined by the restricted partition function $Z_H^N(\alpha_{10}, \alpha_{1,\pm 1}; \beta, Q, \Omega)$. Amplitudes $\alpha_{10}, \alpha_{1,\pm 1}$ of the ordered modes appear as parameters in this restricted partition function, and will have to be evaluated separately.

Standard Gaussian integration is used to evaluate the last integral, which yields, after scaling $\beta' N = \beta$, and provided the Gaussian conditions hold - for $l \geq 2$, $\frac{\beta' \lambda_{lm}}{2} + \frac{\eta}{Q} = \frac{\beta'}{2l(l+1)} + \frac{\eta}{Q} > 0$ - the partition function

$$Z_H^N(\alpha_{10}, \alpha) \propto \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \; \exp\left\{ N \left[\begin{array}{c} \eta - \frac{\beta'}{2}\Omega C \alpha_{10} \\ -\left(\frac{\beta'}{4} + \frac{\eta}{Q}\right) \sum_{m=-1}^{1} \alpha_{1m}^2 \\ -\frac{1}{2N} \sum_{l=2} \sum_m \ln\left(\frac{N\eta}{Q} + \frac{\beta' N}{2}\lambda_{lm} \right) \end{array} \right] \right\}$$

where the free energy per site evaluated at the most probable macrostate is $-\frac{1}{\beta'}$ $F(\eta(\beta'), Q, \beta')$ with

$F(\eta(\beta'), Q, \beta') = \eta(\beta') \left[1 - \frac{1}{Q} \sum_{m=-1}^{1} \alpha_{1m}^2 \right] - \frac{\beta'}{4} \sum_{m=-1}^{1} \alpha_{1m}^2 - \frac{\beta'}{2}\Omega C \alpha_{10} - \frac{1}{2N} \sum_{l=2} \sum_m \ln\left(\frac{N\eta}{Q} + \frac{\beta' N}{2}\lambda_{lm} \right).$

4.2 Planck's theorem, Saddle points and the Themodynamic limit

Provided that the saddle point $\eta(\beta')$ can be determined at given inverse temperature β', Planck's theorem states that the thermodynamically stable (most probable) macrostate is given by the maximum of the expression $F(\eta(\beta'), Q, \beta')$. At positive temperatures, the structure of this expression where it concerns the ground modes α_{1m}, namely, $\chi(\alpha_{10}, \alpha_{1,\pm 1}; \beta, Q, \Omega) = \eta(\beta') \left[1 - \frac{1}{Q} \sum_{m=-1}^{1} \alpha_{1m}^2 \right] - \left[\frac{\beta'}{4} \sum_{m=-1}^{1} \alpha_{1m}^2 + \frac{\beta'}{2}\Omega C \alpha_{10} \right]$, and the fact that the saddle point $\eta(\beta')$ must be positive, suggests that for any positive value of the saddle point, the expression χ and therefore $F(\eta(\beta'), Q, \beta')$ is maximized by $\sum_{m=-1}^{1} \alpha_{1m}^2 = 0$ for all $\beta' > 0$

when planetary spin Ω is small, and by $\alpha_{10} < 0$ for large $\beta' > 0$ when planetary spin Ω is large. At negative temperatures, we expect to find a finite critical point where the two opposing parts of χ are balanced. We proved that these heuristic expectations are valid by solving the restricted partition function in closed form using the method of steepest descent in [21].

The saddle point condition gives one equation for the determination of four variables η, α_{1m} in terms of inverse temperature β' and relative enstrophy Q,

$$0 = \frac{\partial F}{\partial \eta} = \left(1 - \frac{1}{Q}\sum_{m=-1}^{1} \alpha_{1m}^2\right) - \frac{1}{2NQ}\sum_{l=2}\sum_{m}\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2}\lambda_{lm}\right)^{-1} \tag{2}$$

where $\eta = \eta(\beta')$ is taken to be the value of the saddle point. Note that it does not depend on the planetary spin rate $\Omega > 0$. We note in passing that the same equation holds in the $\Omega = 0$ case. There are two natural subcases for the saddle point condition, namely, (A) the disordered phase (for $|T'| \gg 1$) where equation (2) has finite solution $\eta(\beta') > 0$, and $\alpha_{1m} = 0$ for $m = -1, 0, 1$; and (B) the ordered or condensed phase (for $|T'| \ll 1$) where equation (2) has finite solution $\eta(\beta') > 0$ only when $\alpha_{1m} \neq 0$ for some m. In case (A) solved in [21], there is no need to invoke additional equations of state as the amplitudes $\alpha_{1m} = 0$ for $m = -1, 0, 1$.

Case (B) requires three more conditions to determine the three amplitudes α_{1m} and the saddle point $\eta(\beta') > 0$. They are provided by equations of state (or Planck's theorem) for the condensed phase (which do not hold in the disordered phase):

$$0 = \frac{\partial F}{\partial \alpha_{10}} = -\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)\alpha_{10} - \frac{\beta'}{2}\Omega C \tag{3}$$

$$0 = \frac{\partial F}{\partial \alpha_{1,\pm 1}} = -\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)\alpha_{1,\pm 1}. \tag{4}$$

Thus, a coupled system of four algebraic equations (2), (3), (4) determines four unknowns in terms of the planetary spin $\Omega > 0$, the relative enstrophy $Q > 0$ and the scaled inverse temperature β'. The last two equations of state for $\alpha_{1,\pm 1}$ implies that either $\alpha_{1,\pm 1} = 0$ or $\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right) = 0$. The first equation of state differs from the other two; this represents reduction of the $SO(3)$ symmetry that existed in the $\Omega = 0$ case to S^1 symmetry in the case of nonzero planetary spin. Together these three equations of state imply that when $\Omega > 0$, the only possible solution is without tilt,

$$\alpha_{10} = -\frac{\beta'\Omega C}{2}\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)^{-1} \neq 0, \tag{5}$$

$$\alpha_{1,\pm 1} = 0.$$

These values of α_{lm} will be substituted back into the saddle point condition (2) to yield a single equation solved in [21]. The Gaussian conditions imply that for $l > 1$, $\frac{\beta'}{2l(l+1)} + \frac{\eta(\beta')}{Q} > 0$.

The critical temperature can be obtained from the saddle point condition: (A) in the disordered phase at large $|T|$,

$$1 = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} \tag{6}$$

where the large N limit on the RHS is well-defined and finite for any finite $|\beta'|$ provided $\eta(\beta') \geq \eta^* = \frac{|\beta'|Q}{4} > 0$. The corresponding expressions have well-defined positive limits, i.e., for all negative and finite β',

$\lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(-\frac{\beta'}{4} + \frac{\beta'}{2l(l+1)} \right)^{-1} < \infty$,

and for all positive and finite β',

$\lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\beta'}{4} - \frac{\beta'}{2l(l+1)} \right)^{-1} < \infty$.

And (B) in the ordered phase at small $|T|$,

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} \tag{7}$$

where a similar argument proves that the RHS is well-defined and finite provided $\eta(\beta') \geq \eta^*$. This proves that the thermodynamic or continuum limit of the spherical Heisenberg-Ising model H_H^N is well-defined for all negative temperatures. We show that this thermodynamic limit exists for all positive temperatures as well in [21].

The large $|T|$ or small $|\beta'|$ saddle point condition in case (A),

$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} = Q$,

can be solved and has the property that $\eta(\beta') \searrow 1$ as $|\beta'| \to 0$. In case (B), when $|\beta'|$ is large, we will need to discuss (i) $\beta' < 0$ and (ii) $\beta' > 0$ separately- the details of which can be found in [21] - to arrive at

Remark 1: *Since the extreme saddle point $\eta^* = -\frac{\beta' Q}{4}$ satisfies the saddle point conditions (6) and (7) only at the single value of the temperature $T_c^{'} < 0$ that separates the disordered phase from the condensed phase, but not at other $T < 0$, the usual phenomenon known as, sticking of the saddle point in the ordered phase, does not hold here. A more appropriate label for this new saddle point behaviour seen in the spherical-Heisenberg-Ising models for barotropic flows on a rotating sphere, is jumping and reflection of the saddle point at the negative critical point. Indeed the proof above shows that, for all $\Omega > 0$ and $Q > 0$, and for all $\beta' < \beta_c'(Q) < 0$, the saddle point $\eta(\beta') \geq -\frac{\beta'\Omega C\sqrt{Q}}{4} - \frac{\beta' Q}{4} > \eta^*$.*

We summarize the main results in the theorem:

Theorem 1: (A) *For all spin rate $\Omega > 0$ and relative enstrophy $Q > 0$, the quantity $\beta_c'(Q,N) = \frac{1}{QN} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\lambda_{lm} - \frac{1}{2}\right)^{-1} < 0$ has a well-defined and finite limit, called the critical inverse temperature, $\beta_c'(Q) = \lim_{N\to\infty} \beta_c'(Q,N) > -\infty$, that is independent of the rate of spin Ω.*

(B) Moreover, the thermodynamic limit exists for the spherical Heisenberg-Ising models H_H^N in the sense that for any $Q > 0$ and $\Omega > 0$, the saddle point conditions,

$$1 = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_m \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1}$$

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_m \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1},$$

are well-defined and finite, and the saddle point satisfies the condition $\eta(\beta') \geq \eta^* = -\frac{\beta' Q}{4} > 0$ for all $\beta' < 0$.

(C) For all $\Omega > 0$ and $Q > 0$, and for all $\beta' < \beta_c'(Q) < 0$, the ordered phase takes the form of the tiltless ($\alpha_{1,\pm 1} = 0$) ground mode $\alpha_{10}(\beta', \Omega, Q)\psi_{10}$ with amplitude $\alpha_{10} = -\frac{\beta'\Omega C}{2}\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)^{-1} > 0$, which implies that it is aligned with the rotation $\Omega > 0$ (super-rotating) and is linear in Ω.

(D) For spin Ω large enough, there is a positive critical temperature for transition to counter-rotating organized state at low positive temperatures (or equivalently very low energy).

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