This lecture is a review for test2 which is based only on the material in chapter 3 of the textbook. It will be the basis of the webex meeting this tuesday 2-4pm for which you should read this. Webex meetings will be conducted without camera, only audio (mike muted on entry until in use), txt and whiteboard if necessary to conserve bandwidth. The grader and I will do our best given bandwidth and access of RPI webex accounts to answer your questions as we discuss the topics below. In the event that webEX fails we will rely on emails and webpages to review for test 2. If webEX continues to be deficient we will move to other platforms like skype.

1 The main topics are roughly divided into the sections of chapter 3:

1.1 (I) Duality

Let V be a finite dimensional vector space over field F. The dual space V' is defined to be the set of all linear functionals $l : V \to F$. Make sure you know how to show that this set is a vector space of the same dimension as V.

The dualbasis: given a basis $B_V = \{v_1, ..., v_n\}$ of V, let $B_{V'} = \{\varphi_j \in V', j = 1, ..., n|\ \varphi_j(v_k) = \delta_{jk}\}$.Show that $B_{V'}$ is a basis for V'. Another dual object: Let

$T : V \to W$ be a linear map between vector spaces V, W. The dual $T' : W' \to V'$ is defined by $T'(\phi)(v) = \phi(Tv)$ for all $v \in V$. Know why T' is a linear map? Construct or lookup the proof.

1.2 (II) Matrices

The second topic is the representation of a linear map $T : V \to W$ as m by n matrices $M(T; B_V, B_W)$ with entries in the field F; here $\dim V = n$ and $\dim W = m$,and consequences for duality. Recall that the columns of $M(T; B_V, B_W)$ are precisely $Tv_j = \sum_{k=1}^{m} M_{kj} w_k$ where $B_W = \{w_1, ..., w_m\}$. Go through the dis-

cussion in the text showing that $M(T'; B_{W'}, B_{V'}) = [M(T; B_V, B_W)]^T$ where $B_{V'}$ is dual basis to B_V and $B_{W'}$ is dual to B_W. Check your understanding by constructing the degenerate matrix $M(l)$ of a linear functional $l : V \to F$ where $\dim V = n$.

1.3 (III) Fundamental thm of Linear Algebra

One key result in this chapter is the Fundamental thm of Linear Algebra - it has many applications as students in this course have already seen in the homeworks, as well as several direct applications in the chapter including the next topic.

One version states: Let $T : V \to W$ be a linear map between finite dimensional vector spaces over field F, where $\dim V = n$ and $\dim W = m$. Then

$$\begin{aligned} \dim V &= nullity(T) + rank(T) & (1)\\ nullity(T) &= \dim(nullspace(T)) & (2)\\ rank(T) &= \dim(Range(T)) & (3) \end{aligned}$$

Using the fact that any subspace $U < V$ can be written as $U = nullspace(T)$

for some T, and for $U < V$, there is a $X < V$ such that $V = U \oplus X$, we can expand the applications of this theorem.

Students should try to restate this result in the context of dual spaces and dual maps. We will discuss this during webex meeting on tuesday 2-4 pm.

1.4 (IV) Row rank equals column rank

If there is one result that should be kept in mind from chapter 3, it concerns the theorem: For any rectangular matrx $M_{m\times n}$,the column rank equals the row rank. The column rank is usually known as the rank of M,and is equal to the $\dim(Range(M))$ = max number of linearly independent column vectors of M. Likewise, the row rank is the $\dim(span(row$ vectors of $M))$. This remarkable

result required the full machinery of duality for its formulation and proof. One part of this theory has to do with the annihilator of a set or a subspace $W < V$: $W^0 = \{l \in V' | l(w) = 0,\, w \in W\}$.

You should carefully go through the construction and calculation of the annihilators of nullspaces and ranges of linear maps $T : V \to W$. Especially for simple examples such as $Tv = 3v$.

So for $T : (x, y) = (x-y, y)$ say, find (i) $nullspace(T)$, $Range(T)$, $nullspace(T')$, $Range(T')$,and (ii) their respective annihilators. During Mar 24 2020 webex meeting (join by typing my rcsid: limc) we can discuss solutions to these questions.

After that students should try using the FTLA in the context of these annihilators: towards that aim, you need to review what parent vector space, the annihilator subspace are part of. This is crucial for the application of the FTLA. For example, since $N(T) < V$, its annihilator $(N(T))^0 = \{l \in V' | \; l(n) = 0, n \in N(T)\}$ is a subspace of the dual space V'. Similarly, the $Range(T) < W$ implies that $(Range(T))^0 = \{l \in W' | \; l(r) = 0, r \in Range(T)\} < W'$.

You should review all the hwks assigned in this chapter in preparation for test2. *Bon*Chance.