LA20 *Notes* on Other applications of Product and Quotient Spaces, and Duality:

1. We define a new type of product for vector spaces U and V over the field F :

Let us define a bilinear map in $BL(U \times V, W)$ to be $T : U \times V \to W$ as follows:

$$T(au_1 + bu_2, v) = aT(u_1, v) + bT(u_2, v); T(u, av_1 + bv_2) = aT(u, v_1) + bT(u, v_2)$$

Then consider the bilinear forms $BL(U \times V, F)$ as a special case of bilinear maps. We show that $BL(U \times V, F)$ is a vector space over scalars F: let l_1 and l_2 be in $BL(U \times V, F)$ and $\alpha \in F$, then with additive identity $0(u, v) = 0 \in F$, and noting that $(\alpha l_1 + l_2)(u, v) = \alpha l_1(u, v) + l_2(u, v) \in F$, we have obtained the result.

Next, consider the dual vector space $[BL(U \times V, F)]'$ which is a vector space over F consisting of the duals l' of bilinear forms $l \in BL(U \times V, F)$. We will show that $[BL(U \times V, F)]'$ consists of all $l' = u \otimes v$ (where this is just a symbol for now signaling that l' is dependent somehow on the ordered pair $(u, v) \in U \times V$. And $u \otimes v$ acts on arbitrary $l \in BL(U \times V, F)$ by:

$$(u \otimes v)(l) = l(u, v) \in F.$$

The additive identity in $[BL(U \times V, F)]'$ is $0' = 0 \otimes 0$ since $(0 \otimes 0) + (u \otimes v) = (0 + u) \otimes (0 + v) = (u \otimes v)$ for all $(u \otimes v) \in [BL(U \times V, F)]'$. Scalar multiplication in $[BL(U \times V, F)]'$ is defined by: $[\beta(u \otimes v)](l) = \beta l(u, v) \in F$.

Lets check that this action of l' is linear: for $\alpha \in F$, and l_1, l_2 in $BL(U \times V, F)$,we have

$$\begin{aligned}(u \otimes v)(\alpha l_1 + l_2) &= (\alpha l_1 + l_2)(u, v) = \alpha l_1(u, v) + l_2(u, v) \\ &= \alpha(u \otimes v)(l_1) + (u \otimes v)(l_2).\end{aligned}$$

We will call $[BL(U \times V, F)]' = U \otimes V$, the tensor product of U and V. It is a finite dimensional vector space if U and V are finite dimensional. A basis for $U \otimes V$ is $\{u_j \otimes v_k |\ B_U = \{u_j,\ j = 1, ..., n\}; B_V = \{v_k |\ k = 1, ..., m\}$. This proves that $\dim U \otimes V = \dim U \times \dim V$.

2. Extra questions on this topic for homework 7, for discussion on Tuesday after spring break, and due Friday after test 2 to grader.

Q1: Find a basis for $BL(U \times V, F)$.

Q2: Consider linear maps $T_1 : U_1 \to V_1$ and $T_2 : U_2 \to V_2$ where all vector spaces are finite dimensional over the same field F. Show that $\Phi : U_1 \times U_2 \to$

$V_1 \otimes V_2$ defined by $\Phi(u_1, u_2) = T_1u_1 \otimes T_2u_2 \in ReT_1 \otimes ReT_2$ where Re denotes range, is a bilinear map.

Q3: Given a bilinear map $\beta \in BL(X \times Y, W)$, define its lift $\alpha_\beta \in L(X \otimes Y, W)$ to be the map such that $\alpha_\beta(x \otimes y) = \beta(x, y)$. Show that α_β is a linear map from the tensor product $X \otimes Y$ to W.

Q4: Lift Φ in Q2 to a map $\Psi : U_1 \otimes U_2 \to V_1 \otimes V_2$ by the procedure in Q3. Show that Ψ is a linear map from tensor product $U_1 \otimes U_2$ to tensor product $V_1 \otimes V_2$. We call $\Psi = T_1 \otimes T_2 : U_1 \otimes U_2 \to V_1 \otimes V$, the tensor product of linear maps T_1 and T_2.