GENERAL JACOBI COORDINATES AND HERMAN RESONANCE FOR SOME NON-HELIOCENTRIC CELESTIAL N-BODY PROBLEMS

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ABSTRACT

The general Jacobi symplectic variables, generated by a combinatorial algorithm from the full binary tree $T(N)$, are used to formulate some non-heliocentric gravitational N-body problems in perturbation form. The resulting uncoupled term H_U for $(N-1)$ independent Keplerian motions and the perturbation term H_P are both explicitly dependent on the partial ordering induced by the tree $T(N)$. This leads to suitable conditions on separations of the N bodies for the perturbation to be small. We prove the Herman resonance for a new approximation of the 5 body problem. Full details of the derivations of the perturbation form and Herman resonance are given only in the case of five bodies using the caterpillar binary tree $T_c(5)$.

Keywords General Jacobi coordinates perturbation theory · celestial N-body problems · Herman resonances;

1 Introduction

In this note, we introduce the general Jacobi coordinates [9], [6] as efficient symplectic variables for the gravitational N-body problems of N arbitrary masses. Without further assumptions these variables are suitable for numerical computation of the relative dynamics. Under special kinematic conditions, discussed below, these variables can be used to formulate the $N-body$ Hamiltonian H_N for arbitrary masses into perturbation form, meaning $H_N = H_U + H_P$, where H_U is some simple uncoupled term for $N-1$ independent Keplerian motions and H_P is small compared to H_U in some region D of physical space. Some other applications for the general Jacobi coordinates have been reported recently: Katriel et al [4] gave an algorithm based on Jacobi coordinates for constructing highly entangled EPR states in quantum mechanics; Wang [3] have used the general Jacobi coordinates in the context of symplectic capacity to solve certain problems in vortex dynamics; other applications can be found in [7], [8]

A combinatorial algorithm [9], [6] based on full binary trees with N leave, is the essential part of this generalization of the Jacobi symplectic transformations to relative coordinates and partially-ordered centers of masses. In this algorithm, the N leave denotes the N celestial bodies (their absolute positions), and the internal or non-leaf nodes of the binary tree denote the partially ordered list of centers of masses and associated relative coordinates. The last internal node called the root of the binary tree represents the center of mass of the whole system. For example, the Jacobi coordinates used in the three-body problem [5] correspond to the smallest full binary tree with three leaves. This association between arbitrary binarty trees with N leaves and general symplectic transformations for the celestial N-body problems give the general Jacobi coordinates in [9], [6] the flexibility required to formulate a wide range of dynamical $N-body$ problems as perturbations of uncoupled (simpler) problems. In particular, it allows for more than one way to set up a given celestial N-body problem in perturbation form. Indeed, there are as many ways, in principle, as there are distinct

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full binary trees with N leaves. Moreover, it will be clear from the discussion below that the combinatorial structure of the particular full binary tree $T(N)$ used to generate the general Jacobi transformations plays an explicit role in the perturbation theory as well as in the Jacobi symplectic integration. It must be emphasized however, that for a given N-body problem, depending on what the mass ratios are, several different binary trees $T(N)$ can be used to reformulate the Hamiltonian into perturbation form. This choice rests mainly on one's aim in performing this calculation in the first place.

In summary of the content of this note, we will first give explicit examples of how to use general Jacobi variables to formulate $N = 5$ dynamics in perturbation form heuristically. Next, for a special case of binary trees $T(5)$, known as the caterpillar tree $T_c(5)$, we give sufficient conditions for the perturbation term $H_P(T_c(5))$ based on $T_c(5)$ to be much smaller than the uncoupled term $H_U(T_c(5))$ for $N-1$ Keplerian motions in (non-planetary) cases of gravitational N-body problems (where the masses are arbitrary) for which there are many distinct length scales separating the masses. Under these conditions, the Jacobi variables are more efficient than otherwise, in the sense that the smaller terms in H_P can now be neglected in the short to medium times. The uncoupled term in this note are typically $N-1$ Keplerian motions that, unlike the planetary problem, are not based on two-body pairs consisting of the sun and each planet, as in Arnold [11]. Under similar conditions, we prove Herman's resonance for a nerw approximation of the $5-bodies$ problem and by extension to the general caterpillar tree $T_c(N)$ on N leaves, the same for $N-bodies$.

These sufficient conditions are induced by the partial ordering of the associated tree $T(N)$. In the particular case of $T_c(N)$, they are as follows: let d_1 to d_{N-1} be the length scales in the initial data, then there is a small parameter $\varepsilon \ll 1$ such that $d_j/d_{j+1} = O(\varepsilon)$. However, these conditions are quite difficult to verify as they require in general the full machinery of KAM theory. Nonetheless there are several such proofs over the past 50 years, mostly for the planetary problem [18], [17], [12], [13],[14], [15], [19], [22].

We recall that Arnold obtained a full proof of KAM-tori for the three-body problem but his proof for the fully N-body heliocentric problem was not completed until 2004 by Fejoz [17] via a method suggested by Herman [18]. In Fejoz's completion of the proof of Arnold's theorem, he used a weaker form of the KAM theory bypassing the full nondegneracy requirements, focussing instead on only the first order resonance, and applying a differentiable version of the KAM theory. More recently, a direct approach to the proof of Arnold's theorem was presented by Pinzari [12], [13]. They base their approach on the Deprit transform [16] which put the N-body planetary system in further reduced form, and thence action variables, that is, reduction via the conserved angular momentum. We will not need to go to the lengths in [17], [12], [13] to achieve the limited aims here. We hope in the future that it may be possible to use the general Jacobi variables in similar proofs of KAM tori, where these Jacobi transformations should be done prior to the subsequent reduction of each of the $N-1$ independent Keplerian motions to Poincare - Delaunay and Deprit variables.

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2 $N = 5$ Body Case: two examples of $T(5)$

Two examples of binary trees with 5 leaves and the general Jacobi transforms that they generate will be given here in full details. In the next section, a third binary tree $T_c(5)$, called the caterpillar with five legs, will be used to perform explicitly the lengthy calculations leading to a perturbation form $H_N = H_U + H_P$, which supports the derivation of a linear secular system and Herman resonances.

2.1 $T(5; (3, 2))$

Let us call the masses of the $N = 5$ case, m_i, $i = 1, ..., 5$. Let us consider the full binary tree $T(5; (3, 2))$ [9] where the labels $(3, 2)$ in the second argument completely specifies the tree involved: the first branching at the root divides the 5 leaves (masses) into two subsets of sizes 3 and 2,and the full binary structure then implies that the level two branchings split the subset of size 3 into sets of two and one leave, and the subset of size 2 into 2 leave, with the last and only level 3 branching splitting the set of two leaves. Next we map the 5 masses onto the leaves: at the root branching, the subset of size 3 consists of m_5 and the pair (m_1, m_2),while the subset of size 2 consists of the pair (m_3, m_4).

The full Hamiltonian function of the 5 bodies Newtonian problem is

$$H_5 \quad = \quad \frac{1}{2}K - U = \frac{1}{2}\sum_{j=1}^{5} m_j v_j^2 - \sum_{j<k}^{5} \frac{m_j m_k}{r_{jk}} \tag{1}$$

$$= \frac{1}{2}\sum_{j=1}^{5}\frac{\mathbf{p}_j^2}{m_j} - \sum_{j<k}^{5}\frac{m_j m_k}{||\mathbf{q}_k - \mathbf{q}_j||} \tag{2}$$

where $\mathbf{q}_j$ are real 3-vectors for the absolute positions of the masses, and $\mathbf{p}_j$ are $real\ 3-vectors$ for their momenta.

The general Jacobi transformation associated with the tree $T(5;(3,2))$,is given by [lim89]

$$\begin{pmatrix} \overrightarrow{Q} \\ \overrightarrow{P} \end{pmatrix} = \mathbf{M}\begin{pmatrix} \overrightarrow{q} \\ \overrightarrow{p} \end{pmatrix} \tag{3}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \tag{4}$$

where the symplectic matrix $\mathbf{M}$ has diagonal blocks given by

$$\mathbf{A} = \begin{bmatrix} -\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ -\frac{\mathbf{m_1}}{\mathbf{m_1+m_2}} & -\frac{\mathbf{m_2}}{\mathbf{m_1+m_2}} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ -\frac{\mathbf{m_1}}{\mathbf{m_1+m_2+m_5}} & -\frac{\mathbf{m_2}}{\mathbf{m_1+m_2+m_5}} & \frac{\mathbf{m_3}}{\mathbf{m_3+m_4}} & \frac{\mathbf{m_4}}{\mathbf{m_3+m_4}} & -\frac{\mathbf{m_5}}{\mathbf{m_1+m_2+m_5}} \\ \frac{\mathbf{m_1}}{\sum_{j=1}^5 \mathbf{m}_j} & \frac{\mathbf{m_2}}{\sum_{j=1}^5 \mathbf{m}_j} & \frac{\mathbf{m_3}}{\sum_{j=1}^5 \mathbf{m}_j} & \frac{\mathbf{m_4}}{\sum_{j=1}^5 \mathbf{m}_j} & \frac{\mathbf{m_5}}{\sum_{j=1}^5 \mathbf{m}_j} \end{bmatrix} \tag{5}$$

$$\mathbf{D} = \begin{bmatrix} \left[\frac{m_1 m_2}{m_1+m_2}\right] & \frac{-\mathbf{1}}{\mathbf{m_1}} & \frac{\mathbf{1}}{\mathbf{m_2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \left[\frac{m_3 m_4}{m_3+m_4}\right] & \mathbf{0} & \mathbf{0} & \frac{-\mathbf{1}}{\mathbf{m_3}} & \frac{\mathbf{1}}{\mathbf{m_4}} & \mathbf{0} \\ \left[\frac{(m_1+m_2)m_5}{m_1+m_2+m_5}\right] & \frac{-\mathbf{1}}{\mathbf{m_1+m_2}} & \frac{-\mathbf{1}}{\mathbf{m_1+m_2}} & \mathbf{0} & \mathbf{0} & \frac{\mathbf{1}}{\mathbf{m_5}} \\ \left[\frac{(m_1+m_2+m_5)(m_3+m_4)}{\sum_{j=1}^5 m_j}\right] & \frac{-\mathbf{1}}{\mathbf{m_1+m_2+m_5}} & \frac{-\mathbf{1}}{\mathbf{m_1+m_2+m_5}} & \frac{\mathbf{1}}{\mathbf{m_3+m_4}} & \frac{\mathbf{1}}{\mathbf{m_3+m_4}} & \frac{-\mathbf{1}}{\mathbf{m_1+m_2+m_5}} \\ \left[\frac{1}{\sum_{j=1}^5 m_j}\right] & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix} \tag{6}$$

Each bold entry in $\mathbf{A}$ and $\mathbf{D}$ represents a real 3 by 3 diagonal matrix; for example,

$$-\mathbf{1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{7}$$

$$\frac{\mathbf{1}}{\mathbf{m_2}} = \begin{bmatrix} m_2^{-1} & 0 & 0 \\ 0 & m_2^{-1} & 0 \\ 0 & 0 & m_2^{-1} \end{bmatrix}. \tag{8}$$

In other words, $\mathbf{A}$ and $\mathbf{D}$ are real 15 by 15 matrices consisting of 5 by 5 entries, each of which is a 3 by 3 diagonal block.

Proposition 1 *After this transformation the new Hamiltonian function representing* 4 *uncoupled Keplerian motions (each of which is a two-cluster motion where the clusters are specified by the branchings in* $T(5;(3,2))$, *is given in relative Jacobi variables* $(\mathbf{Q},\mathbf{P}) = ((\overrightarrow{\mathbf{Q}}_1, ..., \overrightarrow{\mathbf{Q}}_4), (\overrightarrow{\mathbf{P}}_1, .., \overrightarrow{\mathbf{P}}_4))$ *by*

$$H_U = \frac{1}{2}K' - U' = \frac{1}{2}\sum_{i=1}^{4}\frac{\overrightarrow{\mathbf{P}}_i^2}{M_i} \tag{9}$$

$$- \begin{bmatrix} \frac{m_1 m_2}{||\overrightarrow{\mathbf{Q}}_1||} + \frac{m_3 m_4}{||\overrightarrow{\mathbf{Q}}_2||} + \frac{(m_1+m_2)m_5}{||\overrightarrow{\mathbf{Q}}_3||} \\ + \frac{(m_1+m_2+m_5)(m_3+m_4)}{||\overrightarrow{\mathbf{Q}}_4||} \end{bmatrix} \tag{10}$$

The virtual masses M_i *are given by*

$$M_1 = \frac{m_1 m_2}{(m_1+m_2)}, M_2 = \frac{m_3 m_4}{(m_3+m_4)} \tag{11}$$

$$M_3 = \frac{(m_1+m_2)m_5}{(m_1+m_2+m_5)}, M_4 = \frac{(m_1+m_2+m_5)(m_3+m_4)}{\sum_{j=1}^5 m_j} \tag{12}$$

2.2 $T(5;(2,2,1))$

In comparison, by starting with a different full binary tree, for example $T(5;(2,2,1))$ [9] where the root branching splits 5 masses into subsets of size 4 and 1,and then the next branching splits the size 4 subset into 2 and 2 masses, we end up with (as the reader can check) a different symplectic matrix M' which takes the same full Hamiltonian function H_N for 5 masses to a different uncoupled Hamiltonian function for 4 independent Keplerian motions involving a different partially-ordered list of centers of masses:

$$H_U = \frac{1}{2}K' - U' = \frac{1}{2}\sum_{i=1}^{4}\frac{\overrightarrow{\mathbf{P}}_i^2}{M_i} \tag{13}$$

$$- \left[\begin{array}{c} \frac{m_1 m_2}{||\overrightarrow{\mathbf{Q}}_1||} + \frac{m_3 m_4}{||\overrightarrow{\mathbf{Q}}_2||} + \frac{(m_1+m_2)(m_3+m_4)}{||\overrightarrow{\mathbf{Q}}_3||} \\ + \frac{(m_1+m_2+m_3+m_4)m_5}{||\overrightarrow{\mathbf{Q}}_4||} \end{array} \right] \tag{14}$$

Here the virtual masses M_i are given by

$$M_1 = \frac{m_1 m_2}{(m_1+m_2)}, M_2 = \frac{m_3 m_4}{(m_3+m_4)} \tag{15}$$

$$M_3 = \frac{(m_1+m_2)(m_3+m_4)}{(m_1+m_2+m_3+m_4)}, M_4 = \frac{(m_1+m_2+m_3+m_4)m_5}{\sum_{j=1}^{5} m_j} \tag{16}$$

What is significant is the result from [9], [6]:

Theorem 2 *For each of the Catalan number $C(5)$ of distinct full binary trees with 5 leaves, and for any positive masses, m_i, $i = 1, ..., 5$, there belongs a symplectic matrix M of the above A, D blocks type, which transforms the original $(\mathbf{q}, \mathbf{p})$ in H_N to its own general Jacobi variables $(\mathbf{Q}, \mathbf{P})$. Each of this transformation yields a different uncoupled Hamiltonian function H_U on 4 independent Keplerian motions involving its own partially-ordered list of virtual masses and centers of masses. Furthermore, in the general case of $N-bodies$, for each of the Catalan numbers $C(N)$ of full binary trees $T(N)$ with N leave, and for all positive masses, the algorithm [9] transforms symplectically the full $N - body$ Hamiltonian function H_N into perturbation form,*

$$H_N = H_U(T(N)) + H_P(T(N)) \tag{17}$$

where the uncoupled function H_U and the perturbation terms H_P depend on the combinatorial properties of the tree $T(N)$.

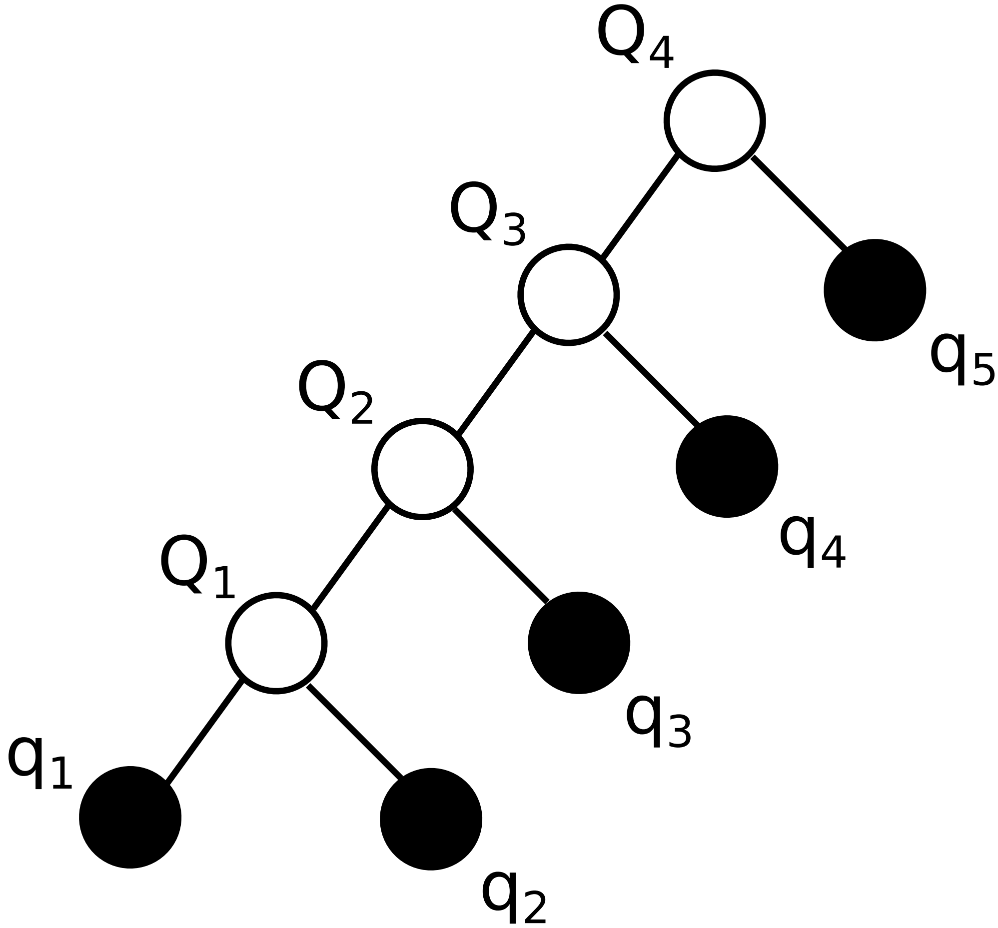
3 Main result: Caterpillar tree $T_c(5)$ and the perturbation terms $H_P(T_c(5))$

In the following discussion, details will be given specifically in the case of the caterpillar binary tree $T_c(5)$. As mentioned in the introduction, the cases of the $N - body$ problem that concerns us are limited by the assumption that there are four distinct length scales in the initial data on these 5 bodies. These four length scales, as will be shown, are natural to $N - body$ dynamics that can be represented on a caterpillar tree.

This tree $T_c(5)$ generates the matrices comprising the diagonal blocks in the symplectic matrix $\mathbf{M}$, according to the algorithm in [6]:

$$\mathbf{A} = \begin{bmatrix} -\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\frac{\mathbf{m}_1}{\mathbf{m}_1+\mathbf{m}_2} & -\frac{\mathbf{m}_2}{\mathbf{m}_1+\mathbf{m}_2} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{\mathbf{m}_1}{\mathbf{m}_1+\mathbf{m}_2+\mathbf{m}_3} & -\frac{\mathbf{m}_2}{\mathbf{m}_1+\mathbf{m}_2+\mathbf{m}_3} & -\frac{\mathbf{m}_3}{\mathbf{m}_1+\mathbf{m}_2+\mathbf{m}_3} & \mathbf{1} & \mathbf{0} \\ -\frac{\mathbf{m}_1}{\sum_{i=1}^{4}\mathbf{m}_i} & -\frac{\mathbf{m}_2}{\sum_{i=1}^{4}\mathbf{m}_i} & -\frac{\mathbf{m}_3}{\sum_{i=1}^{4}\mathbf{m}_i} & -\frac{\mathbf{m}_4}{\sum_{i=1}^{4}\mathbf{m}_i} & \mathbf{1} \\ \frac{\mathbf{m}_1}{\sum_{i=1}^{5}\mathbf{m}_i} & \frac{\mathbf{m}_2}{\sum_{i=1}^{5}\mathbf{m}_i} & \frac{\mathbf{m}_3}{\sum_{i=1}^{5}\mathbf{m}_i} & \frac{\mathbf{m}_4}{\sum_{i=1}^{5}\mathbf{m}_i} & \frac{\mathbf{m}_5}{\sum_{i=1}^{5}\mathbf{m}_i} \end{bmatrix} \tag{18}$$

Figure 1: The leaves of the caterpillar tree $T_c(5)$ correspond to the original positions (momenta) labelled by lowercase q_j. The internal nodes (open circles) correspond to the new symplectic variables labelled by uppercase Q_j. The new coordinates (resp. momenta) are given explicitly by the matrix $A(resp.D)$ in equantion (18) (resp. (19) in terms of the masses.



$$\mathbf{D} = \begin{bmatrix} \left[\frac{m_1 m_2}{m_1+m_2}\right] & \frac{\mathbf{-1}}{\mathbf{m_1}} & \frac{\mathbf{1}}{\mathbf{m_2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \left[\frac{(m_1+m_2)m_3}{m_1+m_2+m_3}\right] & \frac{\mathbf{-1}}{\mathbf{m_1+m_2}} & \frac{\mathbf{-1}}{\mathbf{m_1+m_2}} & \frac{\mathbf{1}}{\mathbf{m_3}} & \mathbf{0} & \mathbf{0} \\ \left[\frac{(\sum_{i=1}^3 m_i)m_4}{\sum_{i=1}^4 m_i}\right] & \frac{\mathbf{-1}}{\mathbf{m_1+m_2+m_3}} & \frac{\mathbf{-1}}{\mathbf{m_1+m_2+m_3}} & \frac{\mathbf{-1}}{\mathbf{m_1+m_2+m_3}} & \frac{\mathbf{1}}{\mathbf{m_3}} & \mathbf{0} \\ \left[\frac{(\sum_{i=1}^4 m_i)m_5}{\sum_{i=1}^5 m_i}\right] & \frac{\mathbf{-1}}{\sum_{i=1}^4 \mathbf{m}_i} & \frac{\mathbf{-1}}{\sum_{i=1}^4 \mathbf{m}_i} & \frac{\mathbf{-1}}{\sum_{i=1}^4 \mathbf{m}_i} & \frac{\mathbf{-1}}{\sum_{i=1}^4 \mathbf{m}_i} & \frac{\mathbf{1}}{\mathbf{m_5}} \\ \left[\frac{1}{\sum_{i=1}^5 m_i}\right] & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix} \tag{19}$$

The uncoupled Hamiltonian function corresponding to $T_c(5)$ is

$$H_U(T_c(5)) = \frac{1}{2}\sum_{j=1}^{4}\frac{\overrightarrow{\mathbf{P}}_j^2}{M_j} - \begin{bmatrix} \frac{m_1 m_2}{||\overrightarrow{\mathbf{Q}}_1||} + \frac{(m_1+m_2)m_3}{||\overrightarrow{\mathbf{Q}}_2||} \\ + \frac{m_4 \sum_{i=1}^3 m_i}{||\overrightarrow{\mathbf{Q}}_3||} + \frac{m_5 \sum_{i=1}^4 m_i}{||\overrightarrow{\mathbf{Q}}_4||} \end{bmatrix}. \tag{20}$$

The inverse transformation for the relative coordinates $\overrightarrow{q}_j - \overrightarrow{q}_k$, $j > k = 1, ..., 5$ in terms of general Jacobi coordinates $\overrightarrow{Q_j}$, $j = 1, ..., 4$ will be needed to expand the perturbation term H_P, and are given by

$$\overrightarrow{q}_2 - \overrightarrow{q}_1 = \overrightarrow{Q_1}, \overrightarrow{q}_3 - \overrightarrow{q}_1 = \overrightarrow{Q_2} + a_1\overrightarrow{Q_1}, \overrightarrow{q}_3 - \overrightarrow{q}_2 = \overrightarrow{Q_2} - b_1\overrightarrow{Q_1}, \tag{21}$$

$$\overrightarrow{q}_4 - \overrightarrow{q}_1 = \overrightarrow{Q_3} + c_2\overrightarrow{Q_2} + a_1\overrightarrow{Q_1}, \overrightarrow{q}_4 - \overrightarrow{q}_2 = \overrightarrow{Q_3} + c_2\overrightarrow{Q_2} - b_1\overrightarrow{Q_1}, \tag{22}$$

$$\begin{aligned}
\overrightarrow{q}_5 - \overrightarrow{q}_1 &= \overrightarrow{Q_4} + c_3\overrightarrow{Q_3} + c_2\overrightarrow{Q_2} + a_1\overrightarrow{Q_1}, &(23)\\
\overrightarrow{q}_5 - \overrightarrow{q}_2 &= \overrightarrow{Q_4} + c_3\overrightarrow{Q_3} + c_2\overrightarrow{Q_2} - b_1\overrightarrow{Q_1}, &(24)\\
\overrightarrow{q}_4 - \overrightarrow{q}_3 &= \overrightarrow{Q_3} - d_2\overrightarrow{Q_2}, \overrightarrow{q}_5 - \overrightarrow{q}_3 = \overrightarrow{Q_4} + c_3\overrightarrow{Q_3} - d_2\overrightarrow{Q_2}, &(25)\\
\overrightarrow{q}_5 - \overrightarrow{q}_4 &= \overrightarrow{Q_4} - d_3\overrightarrow{Q_3}, &(26)
\end{aligned}$$

where

$$\begin{aligned}
a_1\overrightarrow{Q_1} &= \overrightarrow{C}_1 - \overrightarrow{q}_1 = \frac{m_1\overrightarrow{q}_1 + m_2\overrightarrow{q}_2}{m_1+m_2} - \overrightarrow{q}_1 = \frac{m_2}{m_1+m_2}\overrightarrow{Q_1} &(27)\\
b_1\overrightarrow{Q_1} &= \overrightarrow{C}_1 - \overrightarrow{q}_2 = \frac{m_1}{m_1+m_2}\overrightarrow{Q_1}, &(28)
\end{aligned}$$

are the weighted vectors from $\overrightarrow{q}_1$ $(resp.\ \overrightarrow{q}_2)$ to center of mass $\overrightarrow{C}_1$, between m_1 and m_2;and

$$\begin{aligned}
c_2\overrightarrow{Q_2} &= \overrightarrow{C}_2 - \overrightarrow{C}_1 = \frac{(m_1+m_2)\overrightarrow{C}_1 + m_3\overrightarrow{q}_3}{\sum_{i=1}^3 m_i} - \overrightarrow{C}_1 = \frac{m_3}{\sum_{i=1}^3 m_i}\overrightarrow{Q_2} &(29)\\
c_3\overrightarrow{Q_3} &= \overrightarrow{C}_3 - \overrightarrow{C}_2 = \frac{\overrightarrow{C}_2(\sum_{i=1}^3 m_i) + m_4\overrightarrow{q}_4}{\sum_{i=1}^4 m_i} - \overrightarrow{C}_2 = \frac{m_4}{\sum_{i=1}^4 m_i}\overrightarrow{Q_3} &(30)
\end{aligned}$$

are the weighted vector along $\overrightarrow{Q_2}$ $(resp.\overrightarrow{Q_3})$ from $\overrightarrow{C}_1$ to center of mass $\overrightarrow{C}_2$, between (m_1+m_2) and m_3 (resp. $\overrightarrow{C}_2$ to center of mass $\overrightarrow{C}_3$, between $(m_1+m_2+m_3)$ and m_4). Furthermore,

$$\begin{aligned}
d_2\overrightarrow{Q_2} &= \overrightarrow{C}_2 - \overrightarrow{q}_3 = \frac{(m_1+m_2)\overrightarrow{C}_1 + m_3\overrightarrow{q}_3}{\sum_{i=1}^3 m_i} - \overrightarrow{q}_3 = \frac{(m_1+m_2)}{\sum_{i=1}^3 m_i}\overrightarrow{Q_2} &(31)\\
d_3\overrightarrow{Q_3} &= \overrightarrow{C}_3 - \overrightarrow{q}_4 = \frac{\overrightarrow{C}_2(\sum_{i=1}^3 m_i) + m_4\overrightarrow{q}_4}{\sum_{i=1}^4 m_i} - \overrightarrow{q}_4 = \frac{\sum_{i=1}^3 m_i}{\sum_{i=1}^4 m_i}\overrightarrow{Q_3} &(32)
\end{aligned}$$

are the weighted vectors from $\overrightarrow{q}_3$ $(resp.\ \overrightarrow{q}_4)$ to center of mass $\overrightarrow{C}_2$ $(resp.\overrightarrow{C}_3)$.

In terms of the caterpillar tree $T_c(5)$,

$$\begin{aligned}
H_P &= H_N - H_U = -\sum_{j<k}^{5}\frac{m_j m_k}{r_{jk}} &(33)\\
&\quad + \left[\frac{m_1 m_2}{||\overrightarrow{\mathbf{Q}}_1||} + \frac{(m_1+m_2)m_3}{||\overrightarrow{\mathbf{Q}}_2||} + \frac{(\sum_{i=1}^3 m_i)m_4}{||\overrightarrow{\mathbf{Q}}_3||} + \frac{(\sum_{i=1}^4 m_i)m_5}{||\overrightarrow{\mathbf{Q}}_4||}\right] &(34)\\
&= H_1 + H_2; &(35)\\
H_1 &= -\left[\begin{array}{c}\frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}} + \frac{m_1m_4}{r_{14}} + \frac{m_2m_4}{r_{24}} + \frac{m_1m_5}{r_{15}}\\ + \frac{m_2m_5}{r_{25}} + \frac{m_3m_4}{r_{34}} + \frac{m_3m_5}{r_{35}} + \frac{m_4m_5}{r_{45}}\end{array}\right] &(36)\\
&\quad H_2 = \left[\frac{(m_1+m_2)m_3}{||\overrightarrow{\mathbf{Q}}_2||} + \frac{(\sum_{i=1}^3 m_i)m_4}{||\overrightarrow{\mathbf{Q}}_3||} + \frac{(\sum_{i=1}^4 m_i)m_5}{||\overrightarrow{\mathbf{Q}}_4||}\right]. &(37)
\end{aligned}$$

Next, we note that each term in $H_1(T_c(5))$ has the form

$$U_{jk} = \frac{m_j m_k}{r_{13}} = \frac{m_j m_k}{||\overrightarrow{q}_j - \overrightarrow{q}_k||} = \frac{m_j m_k}{||d_{jk}\overrightarrow{Q_4} + c_{jk}\overrightarrow{Q_3} + b_{jk}\overrightarrow{Q_2} + a_{jk}\overrightarrow{Q_1},||} \quad (38)$$

where $a_{jk}, b_{jk}, c_{jk}, d_{jk}$ are in terms of the masses m_i and according to the partial order of the caterpillar tree $T_c(5)$. For example,

$$\begin{aligned}
\overrightarrow{q}_5 - \overrightarrow{q}_3 &= f(\overrightarrow{Q_4}, \overrightarrow{Q_3}, \overrightarrow{Q_2}) = d_{53}\overrightarrow{Q_4} + c_{53}\overrightarrow{Q_3} + b_{53}\overrightarrow{Q_2}, &(39)\\
d_{53} &= 1, |c_{53}| = \frac{m_4}{\sum_{i=1}^4 m_i}, |b_{53}| = \frac{(m_1+m_2)}{\sum_{i=1}^3 m_i}. &(40)
\end{aligned}$$

We also note that the partial order in $T_c(5)$ implies that we can pair the following two expressions

$$\overrightarrow{q}_3 - \overrightarrow{q}_1 = \overrightarrow{Q_2} + a_1\overrightarrow{Q_1}, \overrightarrow{q}_3 - \overrightarrow{q}_2 = \overrightarrow{Q_2} - b_1\overrightarrow{Q_1}, \tag{41}$$

so that,

$$\begin{aligned} |U_{31}+U_{32}| &= \left|\frac{m_3m_1}{||\overrightarrow{q}_3-\overrightarrow{q}_1||}+\frac{m_3m_2}{||\overrightarrow{q}_3-\overrightarrow{q}_2||}\right| && (42)\\ &= \left|m_3\left(\frac{m_1}{||\overrightarrow{Q_2}+a_1\overrightarrow{Q_1}||}+\frac{m_2}{||\overrightarrow{Q_2}-b_1\overrightarrow{Q_1}||}\right)\right| && (43)\\ &= m_3\left|\left(\frac{m_1}{||\overrightarrow{Q_2}+a_1\overrightarrow{Q_1}||}+\frac{m_2}{||\overrightarrow{Q_2}-b_1\overrightarrow{Q_1}||}\right)\right| && (44)\\ &\le m_3(m_1+m_2)\left\{\frac{2}{||\overrightarrow{Q_2}||}\right\} && (45)\end{aligned}$$

since $\frac{1}{||\overrightarrow{Q_2}+a_1\overrightarrow{Q_1}||}$, $\frac{1}{||\overrightarrow{Q_2}-b_1\overrightarrow{Q_1}||}$ are both smaller than $\frac{1}{||\overrightarrow{Q_2}||}$ by the geometry of the triangle $(\overrightarrow{q}_3, \overrightarrow{q}_1, \overrightarrow{q}_2)$,where $\overrightarrow{Q_1}$ is one of the sides and $\overrightarrow{Q_2}+a_1\overrightarrow{Q_1}$, $\overrightarrow{Q_2}-b_1\overrightarrow{Q_1}$ are the remaining sides.
This suggests that we write $U_{31}+U_{32}$ in H_1 as

$$\begin{aligned} U_{31}+U_{32} &= m_3\left(\frac{m_1}{||\overrightarrow{Q_2}+a_1\overrightarrow{Q_1}||}+\frac{m_2}{||\overrightarrow{Q_2}-b_1\overrightarrow{Q_1}||}\right) && (46)\\ &= m_3(m_1+m_2)||\overrightarrow{Q_2}||^{-1}\left(1+...\right) && (47)\end{aligned}$$

where h.o.t. is a power series expansion in the ratio $\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_2}||}$, with coefficients no larger than $\left(\frac{m_1m_2}{m_1+m_2}\right)$. This last claim is shown as follows: we have (similarly for $\frac{m_2}{||\overrightarrow{Q_2}-b_1\overrightarrow{Q_1}||}$)

$$\begin{aligned} &\frac{m_1m_3}{||\overrightarrow{Q_2}+a_1\overrightarrow{Q_1}||} && (48)\\ =\;& m_1m_3\left\|\overrightarrow{Q_2}+a_1\left[(\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}}{||\overrightarrow{Q_2}||})\frac{\overrightarrow{Q_2}}{||\overrightarrow{Q_2}||}+(\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||})\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||}\right]\right\|^{-1} && (49)\\ =\;& m_1m_3\left\|\overrightarrow{Q_2}\left[1+a_1\left(\left(\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}}{||\overrightarrow{Q_2}||^2}\right)\right)\right]+a_1(\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||})\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||}\right\|^{-1} && (50)\\ =\;& m_1m_3\left\{\begin{array}{c}\left\|\overrightarrow{Q_2}\left[1+a_1\left(\left(\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}}{||\overrightarrow{Q_2}||^2}\right)\right)\right]\right\|^2\\ +\left\|a_1(\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||})\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||}\right\|^2\end{array}\right\}^{-1/2} && (51)\\ =\;& m_1m_3\left\{\begin{array}{c}\left\|\overrightarrow{Q_2}\right\|^2\left[1+a_1\left(\left(\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}}{||\overrightarrow{Q_2}||^2}\right)\right)\right]^2\\ +\left\|\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||}\right\|^2\left(a_1\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||}\right)^2\end{array}\right\}^{-1/2} && (52)\\ =\;& m_1m_3\left\{\left\|\overrightarrow{Q_2}\right\|^{-1}Y_{13}\right\}, && (53)\end{aligned}$$

where, to leading order,

$$Y_{13} = \left\{\begin{array}{c}\left[1+a_1\left(\left(\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}}{||\overrightarrow{Q_2}||^2}\right)\right)\right]^2\\ +\left\|\overrightarrow{Q_2}\right\|^{-2}\left\|\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||}\right\|^2\left(a_1\overrightarrow{Q_1}\cdot\frac{\overrightarrow{Q_2}^{\perp}}{||\overrightarrow{Q_2}^{\perp}||}\right)^2\end{array}\right\}^{-1/2} \tag{54}$$

$$= \left\{1+\left(\frac{m_2}{m_1+m_2}\right)\left\{2\alpha_{13}\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_2}||}+(\alpha_{13}^2+\beta_{13})\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_2}\right\|}\right)^2\right\}\right\}^{-1/2} \quad (55)$$

$$= 1-\frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right)\left\{2\alpha_{13}\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_2}||}+(\alpha_{13}^2+\beta_{13})\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_2}\right\|}\right)^2\right\}. \quad (56)$$

where $\alpha_{13}=\cos\left(\overrightarrow{Q_1}\cdot\overrightarrow{Q_2}\right)$ and $\beta_{13}=\cos\left(\overrightarrow{Q_1}\cdot\overrightarrow{Q_2}^{\perp}\right)$.

Combining the leading order term in $-(U_{31}+U_{32})$ in H_1 with the term $m_3(m_1+m_2)||\overrightarrow{Q_2}||^{-1}$ in H_2, we get, to leading order,

$$\begin{aligned} H_P(12,3) &= -m_3(m_1Y_{13}+m_2Y_{23})\left\{\left\|\overrightarrow{Q_2}\right\|^{-1}\right\}+m_3(m_1+m_2)||\overrightarrow{Q_2}||^{-1} && (57)\\ &= m_3||\overrightarrow{Q_2}||^{-1}\left\{(m_1+m_2)-Y_{12,3}\right\} && (58)\\ &= \frac{1}{2}\left(\frac{m_1m_2m_3}{m_1+m_2}\right)||\overrightarrow{Q_2}||^{-1}\left\{2\alpha\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_2}||}+(\alpha^2+\beta)\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_2}\right\|}\right)^2\right\} && (59)\end{aligned}$$

where, to leading order,

$$Y_{12,3}=\left\{\begin{array}{l} m_1\left[1-\frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right)\left\{2\alpha_{13}\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_2}||}+(\alpha_{13}^2+\beta_{13})\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_2}\right\|}\right)^2\right\}\right]\\ +m_2\left[1-\frac{1}{2}\left(\frac{m_1}{m_1+m_2}\right)\left\{2\alpha_{23}\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_2}||}+(\alpha_{23}^2+\beta_{23})\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_2}\right\|}\right)^2\right\}\right]\end{array}\right\}. \quad (60)$$

The same arguments work for the remaining terms U_{jk}, following the partial order of $T_c(5)$. For example, combining the terms involving the internal node in $T_c(5)$ associated with $\overrightarrow{\mathbf{Q}}_3$,and after using

$$\begin{aligned} c_2 &= \frac{m_3}{\sum_{i=1}^3 m_i}, c_3=\frac{m_4}{\sum_{i=1}^4 m_i}, && (61)\\ d_2 &= \frac{(m_1+m_2)}{\sum_{i=1}^3 m_i}, d_3=\frac{\sum_{i=1}^3 m_i}{\sum_{i=1}^4 m_i}; && (62)\\ \overrightarrow{q}_4-\overrightarrow{q}_1 &= \overrightarrow{Q_3}+c_2\overrightarrow{Q_2}+a_1\overrightarrow{Q_1}, && (63)\\ \overrightarrow{q}_4-\overrightarrow{q}_2 &= \overrightarrow{Q_3}+c_2\overrightarrow{Q_2}-b_1\overrightarrow{Q_1}, && (64)\\ \overrightarrow{q}_4-\overrightarrow{q}_3 &= \overrightarrow{Q_3}-d_2\overrightarrow{Q_2}, && (65)\end{aligned}$$

we get, at the leading order, from H_1 and H_2,

$$\begin{aligned} H_P(123,4) &= -(U_{14}+U_{24}+U_{34})+\frac{(\sum_{i=1}^3 m_i)m_4}{||\overrightarrow{\mathbf{Q}}_3||} && (66)\\ &= m_4||\overrightarrow{\mathbf{Q}}_3||^{-1}\left\{-Y_{123,4}+\sum_{i=1}^3 m_i\right\} && (67)\\ &= \frac{1}{2}||\overrightarrow{Q_3}||^{-1}\left\{\begin{array}{c}\left(\frac{m_1m_2m_4}{m_1+m_2}\right)\left\{2\alpha\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_3}||}+(\alpha^2+\beta)\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2\right\}\\ +\frac{2(m_1+m_2)m_3m_4}{\sum_{i=1}^3 m_i}\left\{2A\frac{||\overrightarrow{Q_2}||}{||\overrightarrow{Q_3}||}+(A^2+B)\left(\frac{||\overrightarrow{Q_2}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2\right\}\end{array}\right\} && (68)\end{aligned}$$

where α, β, A and B are cosines of angles between $\overrightarrow{Q_j}$ and $\overrightarrow{Q_k}$, with $j \prec k$ in the partial order of $T_c(5)$, and

$$Y_{123,4} = \left\{ \begin{array}{c} m_1 \left[\begin{array}{c} 1 - \frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right)\left\{ 2\alpha_{14}\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_3}||} + (\alpha_{14}^2 + \beta_{14})\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2 \right\} \\ -\frac{m_3}{2\sum_{i=1}^3 m_i}\left\{ 2A_{14}\frac{||\overrightarrow{Q_2}||}{||\overrightarrow{Q_3}||} + (A_{14}^2 + B_{14})\left(\frac{||\overrightarrow{Q_2}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2 \right\} \end{array} \right] \\ +m_2 \left[\begin{array}{c} 1 - \frac{1}{2}\left(\frac{m_1}{m_1+m_2}\right)\left\{ 2\alpha_{24}\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_3}||} + (\alpha_{24}^2 + \beta_{24})\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2 \right\} \\ -\frac{m_3}{2\sum_{i=1}^3 m_i}\left\{ 2A_{24}\frac{||\overrightarrow{Q_2}||}{||\overrightarrow{Q_3}||} + (A_{24}^2 + B_{24})\left(\frac{||\overrightarrow{Q_2}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2 \right\} \end{array} \right] \\ +m_3 \left[1 - \frac{(m_1+m_2)}{2\sum_{i=1}^3 m_i}\left\{ 2A_{34}\frac{||\overrightarrow{Q_2}||}{||\overrightarrow{Q_3}||} + (A_{34}^2 + B_{34})\left(\frac{||\overrightarrow{Q_2}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2 \right\} + ... \right] \end{array} \right\} \tag{69}$$

Lastly, after using again,

$$\begin{aligned} \overrightarrow{q}_5 - \overrightarrow{q}_1 &= \overrightarrow{Q_4} + c_3\overrightarrow{Q_3} + c_2\overrightarrow{Q_2} + a_1\overrightarrow{Q_1}, && (70) \\ \overrightarrow{q}_5 - \overrightarrow{q}_2 &= \overrightarrow{Q_4} + c_3\overrightarrow{Q_3} + c_2\overrightarrow{Q_2} - b_1\overrightarrow{Q_1}, && (71) \\ \overrightarrow{q}_5 - \overrightarrow{q}_3 &= \overrightarrow{Q_4} + c_3\overrightarrow{Q_3} - d_2\overrightarrow{Q_2}, \overrightarrow{q}_5 - \overrightarrow{q}_4 = \overrightarrow{Q_4} - d_3\overrightarrow{Q_3}, && (72) \\ c_2 &= \frac{m_3}{\sum_{i=1}^3 m_i}, c_3 = \frac{m_4}{\sum_{i=1}^4 m_i}, && (73) \\ d_2 &= \frac{(m_1+m_2)}{\sum_{i=1}^3 m_i}, d_3 = \frac{\sum_{i=1}^3 m_i}{\sum_{i=1}^4 m_i}, && (74) \end{aligned}$$

we have, to leading order,

$$\begin{aligned} H_P(1234,5) &= -(U_{15} + U_{25} + U_{35} + U_{45}) + \frac{(\sum_{i=1}^4 m_i)m_5}{||\overrightarrow{\mathbf{Q}}_4||} && (75) \\ &= \frac{1}{2}||\overrightarrow{Q_4}||^{-1} \left\{ \begin{array}{c} \left(\frac{m_1m_2m_5}{m_1+m_2}\right)\left\{ 2\alpha\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_4}||} + (\alpha^2 + \beta)\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_4}\right\|}\right)^2 \right\} \\ +\frac{2(m_1+m_2)m_3m_5}{\sum_{i=1}^3 m_i}\left\{ 2A\frac{||\overrightarrow{Q_2}||}{||\overrightarrow{Q_4}||} + (A^2 + B)\left(\frac{||\overrightarrow{Q_2}||}{\left\|\overrightarrow{Q_4}\right\|}\right)^2 \right\} \\ +\frac{2(m_1+m_2+m_3)m_4m_5}{\sum_{i=1}^4 m_i}\left\{ 2C\frac{||\overrightarrow{Q_3}||}{||\overrightarrow{Q_4}||} + (C^2 + D)\left(\frac{||\overrightarrow{Q_3}||}{\left\|\overrightarrow{Q_4}\right\|}\right)^2 \right\} \end{array} . \right\} && (76) \end{aligned}$$

Thus, to leading order, the original perturbation is

$$\begin{aligned} H_P(T_c(5)) &= H_P(12,3) + H_P(123,4) + H_P(1234,5) && (77) \\ &= \frac{1}{2}\left(\frac{m_1m_2m_3}{m_1+m_2}\right)||\overrightarrow{Q_2}||^{-1}\left\{ 2\alpha\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_2}||} + (\alpha^2 + \beta)\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_2}\right\|}\right)^2 \right\} && (78) \end{aligned}$$

$$+\frac{1}{2}||\overrightarrow{Q_3}||^{-1}\left\{\begin{array}{c}\left(\frac{m_1m_2m_4}{m_1+m_2}\right)\left\{2\alpha\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_3}||}+(\alpha^2+\beta)\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2\right\}\\+\frac{2(m_1+m_2)m_3m_4}{\sum_{i=1}^3 m_i}\left\{2A\frac{||\overrightarrow{Q_2}||}{||\overrightarrow{Q_3}||}+(A^2+B)\left(\frac{||\overrightarrow{Q_2}||}{\left\|\overrightarrow{Q_3}\right\|}\right)^2\right\}\end{array}\right\} \tag{79}$$

$$+\frac{1}{2}||\overrightarrow{Q_4}||^{-1}\left\{\begin{array}{c}\left(\frac{m_1m_2m_5}{m_1+m_2}\right)\left\{2\alpha\frac{||\overrightarrow{Q_1}||}{||\overrightarrow{Q_4}||}+(\alpha^2+\beta)\left(\frac{||\overrightarrow{Q_1}||}{\left\|\overrightarrow{Q_4}\right\|}\right)^2\right\}\\+\frac{2(m_1+m_2)m_3m_5}{\sum_{i=1}^3 m_i}\left\{2A\frac{||\overrightarrow{Q_2}||}{||\overrightarrow{Q_4}||}+(A^2+B)\left(\frac{||\overrightarrow{Q_2}||}{\left\|\overrightarrow{Q_4}\right\|}\right)^2\right\}\\+\frac{2(m_1+m_2+m_3)m_4m_5}{\sum_{i=1}^4 m_i}\left\{2C\frac{||\overrightarrow{Q_3}||}{||\overrightarrow{Q_4}||}+(C^2+D)\left(\frac{||\overrightarrow{Q_3}||}{\left\|\overrightarrow{Q_4}\right\|}\right)^2\right\}\end{array}.\right\} \tag{80}$$

We have thus completed the proof of :

Theorem 3 *Let the binary tree be $T_c(5)$ be associated with the $N=5$ gravitational system, with arbitrary masses m_i, $i=1,...,5$. Assume that the separations of the masses in the initial data, are given, in terms of the Jacobi coordinates by, $0<d_1=||Q_1||...,d_4=||Q_4||$, such that $d_j/d_{j+1}=O(\varepsilon)$ for some fixed $\varepsilon\ll 1$, and for $j=1,..,3$. Then, under the Jacobi transforms generated by $T_c(5)$, this $5-$body problem has the form $H=H_U(T_c(5))+H_P(T_C(5))$ where*

$$\begin{aligned}H_U(T_c(5)) &= \left\{\frac{1}{2}\frac{\overrightarrow{P}_j^2}{\left(\sum_{i=1}^{j+1}m_i\right)}\right\}-\sum_{j=1}^4\left\{\frac{m_{j+1}\left(\sum_{i=1}^j m_i\right)}{||\overrightarrow{Q}_j||}\right\} &(81)\\ &= \left\{\frac{1}{2}\frac{\overrightarrow{P}_j^2}{\left(\sum_{i=1}^{j+1}m_i\right)}\right\}-\sum_{j=1}^4\left\{\frac{m_{j+1}\left(\sum_{i=1}^j m_i\right)}{d_j}\right\} &(82)\\ H_P(T_C(5)) &= \sum_{j=1}^4 W_j\left\|\overrightarrow{Q_j}\right\|^{-1}\left\{\begin{array}{c}\sum_{k\prec j}\left(A(k)\frac{||\overrightarrow{Q_k}||}{\left\|\overrightarrow{Q_j}\right\|}\right)\\+\left(B(k)\frac{||\overrightarrow{Q_k}||}{\left\|\overrightarrow{Q_j}\right\|}\right)^2\end{array}\right\} &(83)\\ &= \sum_{j=1}^4 W_jd_j^{-1}\left\{\begin{array}{c}\sum_{k\prec j}\left(A(k)\frac{d_k}{d_j}\right)\\+\left(B(k)\frac{d_k}{d_j}\right)^2\end{array}\right\}, &(84)\end{aligned}$$

where the sums in H_P are over internal nodes $k\prec j$, that is, lower than node j in the partial ordering induced by $T_c(5)$, the effective weights W_j are typically of the form $W_3=\frac{(m_1+m_2)m_3m_4}{\sum_{i=1}^3 m_i}$, and the A_k and B_k are uniformly $O(1)$. Furthermore, if all the masses are equal, then the perturbation $H_P(T_C(5))=O(\varepsilon)$ compared to H_U, at $t=0$, and for all $t>0$ in the $12-\dim$ (spatial) region $D=\{(Q_1,...,Q_4)\subset(R^3)^4|\ d_j/d_{j+1}=O(\varepsilon),\ j=1,2,3\}$.

The remaining questions are typically of KAM type (A) does the length scale property persists for all time, (B) if so for what fraction of such initial data, and (C) if not, for how long typically? If there are any proofs they will have to come from KAM and Nekhoroshev theory. Heuristic answers are not difficult to devise: By the separations property which holds at $t=0$, the four Keplerian motions in H_U, involving the two-body problems specified respectively by the Jacobi coordinates, $Q_1$to Q_4 are initially weakly coupled. So, we expect that these four Keplerian motions remain nearly independent for time $T(\varepsilon)$ which is an increasing function $f(1/\epsilon)$.

4 Averaging and linear secular system

In this section, under the same conditions on the length scales as in the Main Theorem, that is, assuming that the $5-bodies$ dynamics is in the region D, we derive the Herman resonances. To derive the first secular system in the $N-body$ case, we follow the process which involves first an averaging step and then linearization at the origin. Unlike the heliocentric planetary case [11], [5], where the origin corresponds to $N-1$ circular coplanar orbits, the origin for linearization in our (non-planetary) problem corresponds to $N-1$ circular but non-coplanar orbits.

To visualize the geometry of the $(N-1)$ Keplerian ellipses implicit in $H_U(T_c(N))$, think of a child's mobile in 3-space: the root r of the tree is the point of attachment of this mobile to the ceiling, the root r is on a straight wire (associated with Jacobi vector $\overrightarrow{Q}_{N-1}$) which has mass m_N at one end and the rest of the mobile at the other end; this is repeated until the last wire (associated with Jacobi coordinate $\overrightarrow{Q}_1$) which is the only wire that has at both ends, original masses, $m_1$and m_2. With this mobile in mind, the $N-1$ ellipses correspond to the orbits of the two-clusters at the ends of each massless wire: clearly elliptical motions require wires that changes in length; however, the limiting circular orbits around which we linearize to derive the first secular system, are consistent with fixed-length wires of the mobile. The image of a fully 3-dimensional mobile now offers the useful view of the $N-1$ circular non-coplanar orbits: the $(j-1)-th$ wire (below the $j-th$ wire in the partial ordering of $T_c(N)$), is in a perfectly circular orbit that lies in the plane whose normal is specified instantaneously by the $j-th$ wire. The instantaneous planes of these circular orbits are now the respective reference planes for the corresponding Keplerian ellipse.

Recall from the Appendix that each of the uncoupled $(N-1)$ Keplerian elliptical motions is given by a vector $(a_j, w_1^j, w_2^j, \widehat{l_j})$, $j=1,...,N-1$. The complex parameters (w_1^j, w_2^j) give the inclination, and perihelion with respect to some reference plane. In the heliocentric planetary case, all the reference planes of the sun-planet Keplerian ellipses are the same one [11]. In our non-planetary problem with arbitrary masses, these $N-1$ reference planes are not only unequal, but they are changing in time, because their normals are specified respectively by the instantaneous values of the $N-1$ Jacobi vectors, $\overrightarrow{Q}_1(t), ...\overrightarrow{Q}_{N-1}(t)$.

Thus, in both the full H_N and the uncoupled term $H_U((a_j, w_1^j, w_2^j, \widehat{l_j}), j=1,...,4)$, the general Jacobi variables $(\overrightarrow{\mathbf{Q}}_j, \overrightarrow{\mathbf{P}}_j)$ are each associated with a $(a_j, w_1^j, w_2^j, \widehat{l_j})$, $j=1,...,4$. In terms of the latter, the full Newtonian equations of motion after reduction by the center of mass $\sum_{i=1}^{5} m_i \overrightarrow{r_i}$ (where $\overrightarrow{r_i}$ are the original absolute positions of the 5 masses) are: for $j=1,..,4$,

$$\frac{d}{dt}a_j = A_j((a_j, w_1^j, w_2^j, \widehat{l_j}), j=1,...,4) \tag{85}$$

$$\frac{d}{dt}w_{1,2}^j = W_{1,2}^j((a_j, w_1^j, w_2^j, \widehat{l_j}), j=1,...,4) \tag{86}$$

$$\frac{d}{dt}\widehat{l_j} = L_j((a_j, w_1^j, w_2^j, \widehat{l_j}), j=1,...,4). \tag{87}$$

They can be put in canonical Hamiltonian form with Hamiltonian function $H_N(T(5))$ which is NOT the original H_N for 5 bodies in eqn (2), but rather the full reduced Hamiltonian function for coupled relative motions. This is given in perturbation form in terms of the general Jacobi variables by

$$H_N((a_j, w_1^j, w_2^j, \widehat{l_j}), j=1,...,4) = H_U(T(5)) + H_P \tag{88}$$

where H_U and H_P will be given below for $T_c(5)$. In terms of the general masses, $M_j(T(N))$ the associated symplectic form is

$$\omega = M_1 dP_1 \wedge dQ_1 + M_2 dP_2 \wedge dQ_2 + M_3 dP_3 \wedge dQ_3 + M_4 dP_4 \wedge dQ_4 \tag{89}$$

by the results in [9].

With abuse of notation, we define the averaged system [2] for $j=1,...,N-1$,

$$\frac{d}{dt}a_j = \frac{1}{(2\pi)^{N-1}}\int_0^{2\pi} d\widehat{l_1}...\int_0^{2\pi} d\widehat{l_{N-1}}A_j \tag{90}$$

$$\frac{d}{dt}w_{1,2}^j = \frac{1}{(2\pi)^{N-1}}\int_0^{2\pi} d\widehat{l_1}...\int_0^{2\pi} d\widehat{l_{N-1}}W_{1,2}^j \tag{91}$$

$$\frac{d}{dt}\widehat{l_j} = \frac{1}{(2\pi)^{N-1}}\int_0^{2\pi} d\widehat{l_1}...\int_0^{2\pi} d\widehat{l_{N-1}}L_j. \tag{92}$$

where the $(a_j, w_1^j, w_2^j, \widehat{l_j})$, $j = 1, ..., 4$ are each associated with the respective general Jacobi variables $(\overrightarrow{\mathbf{Q}}_j, \overrightarrow{\mathbf{P}}_j)$. We will use both systems of variables interchangeably. Applying Proposition 2 in [5] to this system, we find it has a Hamiltonian function given by $(N-1)$ iterated integrals,

$$\overline{H} = \frac{1}{(2\pi)^{N-1}} \int_0^{2\pi} d\widehat{l_1} .... \int_0^{2\pi} d\widehat{l_{N-1}} H_N((a_j, w_1^j, w_2^j, \widehat{l_j}), j = 1 to 4) \quad (93)$$

$$= \overline{H}_U + \overline{H}_P; \quad (94)$$

$$\overline{H}_U = \frac{1}{(2\pi)^{N-1}} \int_0^{2\pi} d\widehat{l_1} .... \int_0^{2\pi} d\widehat{l_{N-1}} H_U(T_c(5)) \quad (95)$$

$$\overline{H}_P = \frac{1}{(2\pi)^{N-1}} \int_0^{2\pi} d\widehat{l_1} .... \int_0^{2\pi} d\widehat{l_{N-1}} H_P \quad (96)$$

where

$$\overline{H}_P = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\widehat{l_1} \int_0^{2\pi} d\widehat{l_2} \overline{K}_P \quad (97)$$

$$\overline{K}_P = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\widehat{l_3} \int_0^{2\pi} d\widehat{l_4} H_P((a_j, w_1^j, w_2^j, \widehat{l_j}), j = 1, ..., 4). \quad (98)$$

By the Main theorem, $\overline{H}_P$ is much smaller than $\overline{H}_U$, in the region D.

In that case, we only have to calculate the quadratic term from $\overline{H}_U$ in the derivation of the linear secular system that leads to Herman resonances. We note the significant point here that $H_U = \frac{1}{2}K' - U'$ represent the 4 uncoupled Keplerian motions. In other words, in terms of the elliptical elements $((a_j, w_1^j, w_2^j, \widehat{l_j})$, $j = 1, ..., 4)$ the Hamiltonian dynamics of H_U conserve (a_j, w_1^j, w_2^j), for each $j = 1, ..., 4$. This is the same as the conservation of angular momentum in each of the 4 uncoupled Keplerian motions. Moreover, since each $\overrightarrow{\mathbf{P}}_j$ is independent of $\widehat{l_k}$ for $k \neq j$,we get the averaged uncoupled kinetic energy

$$\frac{1}{2}\overline{K}' = \sum_{j=1}^{4} \frac{1}{(2\pi)^4} \int_0^{2\pi} d\widehat{l_1} .... \int_0^{2\pi} d\widehat{l_4} \frac{\overrightarrow{\mathbf{P}}_j^2}{2M_j} \quad (99)$$

$$= \sum_{j=1}^{4} \frac{1}{(2\pi)} \int_0^{2\pi} d\widehat{l_j} \frac{\overrightarrow{\mathbf{P}}_j^2}{2M_j} \quad (100)$$

which is a constant.

After averaging the above H with respect to the fast angles on the 4 Keplerian ellipses, the resulting averaged Hamiltonian function is

$$\overline{H} = \overline{H}_U + \overline{H}_P \quad (101)$$

$$\overline{H}_P(T_C(5)) = \frac{1}{(2\pi)^2} \sum_{j=1}^{4} W_j \int_0^{2\pi} d\widehat{l_j} \sum_{k \prec j} \int_0^{2\pi} d\widehat{l_k} \left[\begin{array}{c} \left(A(k) \frac{||\overrightarrow{Q_k}||}{\left\| \overrightarrow{Q_j} \right\|} \right) \\ + \left(B(k) \frac{||\overrightarrow{Q_k}||}{\left\| \overrightarrow{Q_j} \right\|} \right)^2 \end{array} \right]. \quad (102)$$

After dropping constants from averaging the kinetic energy in H_U, each term in $\overline{H}_U$ has the form,

$$V_j(T_c(5)) = \frac{m_{j+1} \left(\sum_{i=1}^{j} m_i \right)}{(2\pi)} \int_0^{2\pi} d\widehat{l_j} \frac{1}{||\overrightarrow{Q}_j||}, \quad (103)$$

This will be calculated by an application of the key Lemma for the case $\lambda = -1/2$, for Newtonian potentials. Since we have shown that $\overline{H}_P$ is smaller than $\overline{H}_U$, we need only consider the quadratic terms in the expansion of $\overline{H}_U$ around the origin, to derive the linear secular system.

We use eqn (134) in the appendix for the symplectic form σ of Keplerian motion on a single ellipse, in order to derive the following symplectic form ω for motions on $(N-1)$ ellipses: to first order,

$$\omega = -2i \sum_{j=1}^{N-1} \overline{M}_j (dg_j \wedge dg_j^* + ds_j \wedge ds_j^*) + ... \tag{104}$$

The coefficients in the symplectic form $\omega(T_c(5))$ for the binary tree $T_c(5)$ are:

$$\overline{M}_1 = \frac{m_1 m_2 \sqrt{a_1}}{\sqrt{m_1+m_2}}, \overline{M}_2 = \frac{2m_3(m_1+m_2)\sqrt{a_2}}{\sqrt{\sum_{i=1}^{3} m_i}}, \tag{105}$$

$$\overline{M}_3 = \frac{2m_4(\sum_{i=1}^{3} m_i)\sqrt{a_3}}{\sqrt{\sum_{i=1}^{4} m_i}}, \overline{M}_4 = \frac{2m_5(\sum_{i=1}^{4} m_i)\sqrt{a_4}}{\sqrt{\sum_{i=1}^{5} m_i}}, \tag{106}$$

where a_j is the semi-major axis for the Keplerian ellipse associated with $\overrightarrow{Q}_j$.

Reverting to the variables, $\overrightarrow{w} = (w^j_{1,2}, j = 1, ..., 4) = ((g_j, s_j), j = 1, ..., 4) \in C^8$, we get the following expansions at the origin which is associated with circular non-coplanar orbits,

$$V_j(\overrightarrow{w}) = V_j(\overrightarrow{0}) + \sum_{\mu,\nu} \frac{\partial^2 V_j}{\partial w_\mu \partial w_\nu^*}(\overrightarrow{0}) w_\mu w_\nu^* + ... \tag{107}$$

Using the symplectic form $\omega(T_c(5))$, we get the following Hamilton's equations, with $\mu = (2j-1), 2j$, $j = 1, ..., 4$

$$\frac{dw_\mu}{dt} = -\frac{i}{2\overline{M}_j} \frac{\partial \overline{H}_U}{\partial w_\mu^*} \tag{108}$$

$$\frac{dw_\mu^*}{dt} = \frac{i}{2\overline{M}_j} \frac{\partial \overline{H}_U}{\partial w_\mu} \tag{109}$$

for the function

$$\overline{H}_U = -\sum_{j=1}^{4} V_j. \tag{110}$$

In terms of the 8 by 8 diagonal matrix $\mathbf{Q}$ with entries $d_\mu = \overline{M}_j$ where $\mu = (2j-1), 2j$ and $j = 1, ..., 4$, the linear secular system is given by

$$\frac{d}{dt}\overrightarrow{w} = \frac{1}{2i} \mathbf{Q}^{-1} \left[\partial^* \partial \overline{H}_U\right] \overrightarrow{w} \tag{111}$$

$$\left[\partial^* \partial \overline{H}_U\right]_{\mu\nu} = \frac{\partial^2 \overline{H}_U}{\partial w_\mu^* \partial w_\nu}(\overrightarrow{0}). \tag{112}$$

4.1 Verification of Herman resonances

We apply the key Lemma [5] to prove a version of Proposition 3 in [5] for the $N = 5$ case. The proof is similar for each of the binary trees, $T(5)$:

Proposition 4 *The trace of* $\mathbf{Q}^{-1}\left[\partial^* \partial \overline{H}_U\right]$ *is zero.*

Proof: We note that

$$tr\mathbf{Q}^{-1}\left[\partial^* \partial \overline{H}_U\right] = -\sum_{j=1}^{4} tr\mathbf{Q}^{-1}\left[\partial^* \partial V_j\right] \tag{113}$$

and for each $j - 1, ..., 4$, with $\mu(l) = 2l-1$, $\nu(l) = 2l$, $l = 1, ..., 4$,

$$tr\mathbf{Q}^{-1}\left[\partial^* \partial V_j\right] = \sum_{l=1}^{4} \frac{1}{\overline{M}_l} \cdot \left(\frac{\partial^2 V_j}{\partial w_{\mu(l)} \partial w_{\mu(l)}^*} + \frac{\partial^2 V_j}{\partial w_{\nu(l)} \partial w_{\nu(l)}^*} \right) (\overrightarrow{0}) \tag{114}$$

Next, note that for each $l = 1, ..., 4$,and j,

$$\Delta_{w_{\mu(l)} w_{\nu(l)}} V_j(\overrightarrow{0}) = \left(\frac{\partial^2 V_j}{\partial w_{\mu(l)} \partial w^*_{\mu(l)}} + \frac{\partial^2 V_j}{\partial w_{\nu(l)} \partial w^*_{\nu(l)}} \right) (\overrightarrow{0}) = 0 \tag{115}$$

because V_j is the average over angle $\widehat{l}_j$ of the function $||\overrightarrow{\mathbf{Q}}_j||^{-1}$. The key Lemma for the case $\lambda = -1/2$, implies that

$$\Delta_{w_{\mu(l)} w_{\nu(l)}} V_j(\overrightarrow{0}) = 0 \tag{116}$$

assuming enough smoothness to pass the Laplacian through the integral over angle$\widehat{l}_j$. This completes the proof.

We are now ready to verify Herman resonances in the $N = 5$ case associated with $T_c(5)$. Again we know that the proofs are similar for all the other binary trees $T(5)$, provided that the averaged perturbation $\overline{H}_P$ can be verified to be small, under suitable assumptions on length scales. We should reasonably expect that the same proofs apply to the general $N - body$ problem in that case. The proof of the following can be found in the $N = 3$ case in [5], and follows from the Proposition above

Theorem 5 *(Herman) The linear secular system can be diagonalized with pure imaginary eigenvalues,* $i\lambda_k$, $k = 1, ..., 8$, *where* $\sum_{k=1}^{8} \lambda_k = 0$. *One of the* $\lambda_k = 0$.

5 Appendix: Transformations for a single ellipse

We summarize here for completeness, the transformations that put the motions along a single ellipse in $3 - space$ into a suitable form [5], [12]. We note here the fact that this part of the work is independent of the number of bodies N,and is in fact independent of the subsequent transformations to $N - 1$ uncoupled Keplerian motions. Whether the general Jacobi variables from a binary tree $T(N)$ or some other approach where the $(N - 1)$ two-body motions consist of the Sun-planet pairs, is used to later formulate H_U $(T(N))$, it is immaterial to the work in this subsection. Complex variables will be used for the elements along an ellipse and to locate the ellipse in $3 - space$. The notation used here is that in [5].

Let $a > 0$ be the semi-major axis of the said ellipse. Let Cartesian 3-space and also velocities (v_x, v_y, v_z) be represented in complex form by $(v_c, v_z) \in C \times R$ where $v_c = v_x + iv_y \in C$. Cartesian coordinates for the two-body problem on a fixed ellipse located in the $x - y$ $plane$ are

$$x = a(\cos u - \varepsilon), y = a\sqrt{1 - \varepsilon^2} \sin u, l = u - \varepsilon \sin u \tag{117}$$

where ε is the eccentricity, u is the eccentric anomaly angle, and l is the mean anomaly angle such that dl/dt is a constant.

To fix the perihelion use the angle ω;thus the complex position of the body is

$$R = (x + iy)e^{i\omega} = ae^{i\omega}(\cos u - \varepsilon + i(1 - \frac{1}{2}\varepsilon^2 + ...) \sin u). \tag{118}$$

Regularizing as $\varepsilon \to 0$, by keeping a and $\widetilde{u} = \omega + u$ constant, the limiting equation is [5]:

$$\frac{R}{a} = e^{i\widetilde{u}} - L - \frac{1}{4} e^{i\widetilde{u}} LL^* + \frac{1}{4} e^{-i\widetilde{u}} L^2 + ... \tag{119}$$

where $L = \varepsilon e^{i\omega}$ and the h.o.t. are monomials in $e^{i\widetilde{u}}, L$ and their complex conjugates. Next, define $\varepsilon = \sin\phi$, $\tau = \tan\frac{\phi}{2}$,and replacing L by $k = \tau e^{i\omega}$, derive the rational expression for the position

$$R = \frac{ae^{i\widetilde{u}}(1 - ke^{-i\widetilde{u}})^2}{1 + kk^*} \tag{120}$$

Now put the ellipse into $3 - space$ by using the inclination $0 \leq \delta \leq \pi$, the longitude of the ascending node, Ω which is the angle that the lline of nodes make with the first reference vector ($called$ x-axis) of the reference plane, and the affine transform

$$R \to \left(\cos^2 \frac{\delta}{2}\right) R + \left(\sin^2 \frac{\delta}{2}\right) R^* \tag{121}$$

followed by multiplying by $e^{i\Omega}$. Introducing

$$s = -i\tan\frac{\delta}{2}e^{i\Omega} \quad (122)$$

$$g = ke^{i\Omega} = \tan\frac{\phi}{2}e^{i\widetilde{\omega}} \quad (123)$$

the above affine transform becomes $R \to (1+ss^*)^{-1}(R+ss^*R^*)$ and after defining $\widetilde{\omega} = \omega + \Omega$,called the longitude of perihelion, and $\widehat{u} = \widetilde{u} + \Omega = u + \widetilde{\omega}$,called the eccentric longitude, they obtained the complex form of the position $(r_c, r_z) \in C \times R$ of the body in $3 - space$:

$$(1+ss^*)(1+gg^*)\frac{r_c}{a} = e^{i\widehat{u}}(1+g^*s-(g+s)e^{-i\widehat{u}}) \times \quad (124)$$

$$(1-g^*s-(g-s)e^{-i\widehat{u}}) \quad (125)$$

$$(1+ss^*)(1+gg^*)\frac{r_z}{a} = -e^{i\widehat{u}}(1-ge^{-i\widehat{u}})^2s^* \quad (126)$$

$$-e^{-i\widehat{u}}(1-g^*e^{i\widehat{u}})^2s. \quad (127)$$

Together with the mean anomaly derivative,

$$(1+gg^*)\frac{dl}{du} = (1-ge^{-i\widehat{u}})(1-g^*e^{i\widehat{u}}), \quad (128)$$

we now have a complete description of the motion along an ellipse in 3-space.

Looking forward to the averaging and linearization steps in the next subsection, we will give the symplectic form $\sigma = d\overrightarrow{v} \wedge d\overrightarrow{r}$ for motion on a single Keplerian ellipse in $3 - space$,in terms of the above elements. As before let $\overrightarrow{r} = (r_c, r_z)$, $\overrightarrow{v} = (v_c, v_z)$ where $r_c = r_x + ir_y$ and $v_c = v_x + iv_y$,and the complex symplectic form is $2\sigma = dv_c \wedge dr_c^* + dv_c^* \wedge dr_c + 2dv_z \wedge dr_z$. To obtain $\overrightarrow{v}$ we follow [5] in using the formula $\upsilon^2 a^3 = \mu$ where $\upsilon = dl/dt$ is the frequency of the Keplerian motion. Thus, $\overrightarrow{v} = \upsilon\left(\frac{dl}{du}\right)^{-1}\frac{d\overrightarrow{r}}{du}$,and we have the equations of motion,

$$(1+ss^*)(1+gg^*)\frac{v_c}{i\nu a}\frac{dl}{du} = e^{i\widehat{u}}(1-g^2e^{-2i\widehat{u}}) + e^{-i\widehat{u}}(1-(g^*)^2e^{2i\widehat{u}})s^2 \quad (129)$$

$$(1+ss^*)(1+gg^*)\frac{v_z}{i\nu a}\frac{dl}{du} = -e^{i\widehat{u}}(1-g^2e^{-2i\widehat{u}})s^* \quad (130)$$

$$+e^{-i\widehat{u}}(1-(g^*)^2e^{2i\widehat{u}})s. \quad (131)$$

We will only need the symplectic form at the origin $s = g = 0$, which gives after fixing $\widehat{u}$,

$$dr_c = -2adg + ..., dr_z = a(-e^{i\widehat{u}}ds^* - e^{-i\widehat{u}}ds) + ... \quad (132)$$

$$dv_c = i\nu a(dg + e^{2i\widehat{u}}dg^*) + ..., dv_z = i\nu a(-e^{i\widehat{u}}ds^* + e^{-i\widehat{u}}ds) + ... \quad (133)$$

$$\sigma = i\sqrt{\mu a}(-2dg \wedge dg^* - 2ds \wedge ds^*) + ... \quad (134)$$

We are now ready to state their key Lemma [5] for a single ellipse in general position, which $\lambda = \frac{1}{2}$ case is relevant later:

Lemma 6 *Let a be a nonnegative real number. Let $E(w_1, w_2)$ be the Keplerian ellipse in 3-space with focus at the origin O, semimajor axis a, and complex coordinates (w_1, w_2). Let B be a point in the complement of the reference circle $E(0, 0)$. Let the average be*

$$D_\lambda(w_1, w_2) = \frac{1}{2\pi}\int_0^{2\pi} ||\overrightarrow{AB}||^{2\lambda} d\widehat{l} \quad (135)$$

where A is point on $E(w_1, w_2)$ with mean longitude $\widehat{l} = l + \widetilde{\omega}$, or sum of the mean anomaly and longitude of perihelion. Then,

$$\Delta D_\lambda|_{w_1=w_2=0} = \frac{1}{2}\lambda(2\lambda+1)a^2 D_{\lambda-1}|_{w_1=w_2=0}. \quad (136)$$

In the complex Laplacian $\Delta = \left(\frac{\partial^2}{\partial w_1 \partial w_1^*} + \frac{\partial^2}{\partial w_2 \partial w_2^*}\right)$, the variables (w_1, w_2) may represent any of the following pairs of elliptic elements, including $(2g, 2s)$, $\left(\frac{\xi_c}{\sqrt{2}}, \frac{\eta_c}{\sqrt{2}}\right)$,

$$(L_c, S_c) = \left(\frac{2(g - g^* s^2)}{(1+ss^*)(1+gg^*)}, \frac{2s(1-gg^*)}{(1+ss^*)(1+gg^*)}\right), and \tag{137}$$

$$(X.Y) = \left(\frac{2g}{\sqrt{1+gg^*}}, \sqrt{\frac{1-gg^*}{1+gg^*}}\frac{2s}{\sqrt{1+ss^*}}\right). \tag{138}$$

The usage of the subscript c is the same as above to denote the horizontal or complex part of a $3-vector$. The vectors $\overrightarrow{S}$, $\overrightarrow{L}$ are respectively, the normalized angular momentum and eccentricity vector, with $||\overrightarrow{S}|| = \sqrt{1-\varepsilon^2}$, $||\overrightarrow{L}|| = \varepsilon$, and $\overrightarrow{S} \cdot \overrightarrow{L} = 0$. They satisfy the property that $\sqrt{\mu a}\overrightarrow{S}$ is the angular momentum where μ is the gravitational constant, and $\overrightarrow{L}$ points towards the perihelion of the ellipse. Thus, $\overrightarrow{\xi}$ and $\overrightarrow{\eta}$ are the Souriau vectors defined by $\overrightarrow{\xi} = \overrightarrow{S} + \overrightarrow{L}$ and $\overrightarrow{\eta} = \overrightarrow{S} - \overrightarrow{L}$,with $||\overrightarrow{\xi}|| = ||\overrightarrow{\eta}|| = 1$.

Lastly, the real variables X and Y have the following property: let $I_l = \sqrt{ua}$ be the conjugate to the mean anomaly angle l; then the real and imaginary parts of $\sqrt{I_l}X = x_1 + ix_2$ and $\sqrt{I_l}Y = y_1 + iy_2$ are symplectic variables, known as the Poincare variables, $(I_l, \widehat{l}, x_2, x_1, y_1, -y_2)$. The canonical symplectic form is $dI_l \wedge d\widehat{l} + dx_2 \wedge dx_1 + dy_2 \wedge dy_1$.

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