Discontinuous Galerkin Methods with Optimal L^2 Accuracy for One Dimensional Linear PDEs with High Order Spatial Derivatives

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Abstract

In this paper, we formulate and analyze discontinuous Galerkin (DG) methods to solve several partial differential equations (PDEs) with high order spatial derivatives, including the heat equation, a third order wave equation, a fourth order equation and the linear Schrödinger equation in one dimension. Following the idea of local DG (LDG) methods, we first rewrite each PDE into its first order form and then apply a general DG formulation. The numerical fluxes are introduced as linear combinations of average values of fluxes, and jumps of the solution as well as the auxiliary variables at cell interfaces. The main focus of the present work is to identify a sub-family of the numerical fluxes by choosing the coefficients in the linear combinations, so the solution and some auxiliary variables of the proposed DG methods are optimally accurate in the L^2 norm. In our analysis, one key component is to design some special projection operator(s), tailored for each choice of numerical fluxes in the sub-family, to eliminate those terms at cell interfaces that would otherwise contribute to the sub-optimality of the error estimates. Our theoretical findings are validated by a set of numerical examples.

Keywords: Heat equation; Schrödinger equation; High order wave equation; Discontinuous Galerkin method; Numerical flux; Error estimate.

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1 Introduction

In this paper, we propose discontinuous Galerkin (DG) methods with optimal accuracy for solving several partial differential equations (PDEs) with high order spatial derivatives in one dimension. They include:

- The even order equations
  - the heat equation
$$u_t - u_{xx} = 0; \tag{1.1}$$
  - the fourth order equation
$$u_t + u_{xxxx} = 0; \tag{1.2}$$
  - the equation of an arbitrary even order,
$$u_t + (-1)^{\frac{n}{2}} u_x^{(n)} = 0, \tag{1.3}$$
    with n being any positive even integer. Here $u_x^{(n)}$ denotes the n-th derivative of u with respect to x.
- The third order wave equation
$$u_t + u_{xxx} = 0; \tag{1.4}$$
- The linear Schrödinger equation
$$iu_t + u_{xx} = 0. \tag{1.5}$$

The boundary conditions are assumed to be periodic. These equations provide classical mathematical models for many important physical and engineering applications. The heat equation (1.1) models the heat conduction. The third order wave equation belongs to the KdV-type equations, which describe the propagation of waves in a variety of dispersive media [3]. The fourth order problem (1.2) has wide applications in modeling of thin beams and plates, strain gradient elasticity, and phase separation in binary mixtures [20]. The last equation we consider is the linear Schrödinger equation which has broad applications in fluid dynamics, nonlinear optics, and plasma physics [4, 18].

DG methods are a class of finite element methods using a completely discontinuous piecewise polynomial space for the numerical solution and test functions. The first DG method was introduced in 1973 by Reed and Hill [26] for the linear neutron transport equation. It was then developed for time-dependent nonlinear hyperbolic conservation laws, coupled with high order Runge-Kutta time discretizations, by Cockburn et al. in [15, 14, 13, 12, 17]. DG methods have grown their popularity over the past few decades in many applications due to their flexibility with meshing and local approximations, their compactness and high parallel efficiency, their excellent dispersion property in wave simulations, and their suitability for various types of differential equations (see, e.g. [24, 27]). Particularly the methods find their success in solving time-dependent PDEs with high order spatial derivatives, with several ideas proposed, such as the penalty methods [21, 2] that add penalty terms at cell interfaces for numerical stability; the local DG (LDG) methods [1, 16, 31, 20] that are formulated based on the first order form of the equations by introducing auxiliary variables; the hybrid DG (HDG) methods [11, 8, 7, 19] that, in addition to working with the first order form as in LDG methods, also include the trace of some variables on mesh skeletons as the additional unknowns in order to create opportunity for the ultimate implementation efficiency; the ultra-weak DG (UWDG) methods [10] that are based on repeated applications of integration by parts with all spatial derivatives moved to the test function in the weak formulation; the direct DG (DDG) methods [25] that are based on a more standard weak formulation of a second order diffusive operator; the conservative DG methods [5] that are based on certain weak formulation derived from repeated integration by parts for the dispersive term and a globally defined projection to preserve the energy for the KdV equation. All the methods mentioned above except the penalty type depend on the suitable design of one (such as in DDG methods) or multiple numerical fluxes (e.g. in LDG, HDG, and UWDG methods) in order to achieve numerical stability, (sub-)optimal accuracy of one (e.g. in DDG and UWDG methods) or more (e.g. in LDG and HDG methods) unknowns, and even implementation efficiency (e.g. in LDG and HDG methods).

In this work, we design DG methods for solving the PDEs (1.1)-(1.5) with high order spatial derivatives. Just as in LDG methods, we start with the first order form of each PDE, and apply a general DG formulation. The numerical fluxes are introduced as some linear combinations of average values of fluxes and jumps of the solution as well as the auxiliary variables at cell interfaces, and they involve a set of parameters as the expansion coefficients. Standard LDG methods can be obtained if one takes special values of these parameters to ensure that all the auxiliary variables can be expressed locally in terms of the original unknown. Instead of requiring such local elimination property, we here identify a sub-family of these parameters so that the respective DG methods are optimal in accuracy for the original unknown and also for some auxiliary variables. The LDG methods in [16, 6, 31, 32, 29, 20, 30] for solving equations (1.1)-(1.5) are special cases of what proposed here. Similar numerical fluxes as well as a special sub-family (termed $\alpha\beta$-fluxes) are investigated in [9] to solve the one-dimensional two-way wave problem, and they lead to a class of L^2 stable and optimally accurate DG methods.

The optimal error estimates of our proposed methods reply on two ingredients. One is the energy equations related to numerical stability and the other is the projection operators that measure the approximation property of the discrete spaces and ensure the optimality of the accuracy. For PDEs with high order spatial derivatives, it was discovered in [20, 30] that, to prove optimal accuracy, more than one energy equation may be needed in the presence of the auxiliary variables within the first order forms. These energy equations in general are not trivial to find. Fortunately the stability analyses for LDG methods have partially addressed this aspect for the even order PDEs in [20] (see Remark 3.6) and for the third order wave equation in [30], and for the linear Schrödinger equation in [30] (see Remark 5.2). As for the projection operators, we will follow [9] and work with some similar type of projection operators that are tailored for each choice of numerical fluxes in the identified special sub-family, to eliminate those terms at cell interfaces that would otherwise contribute to the sub-optimality of the error estimates.

For the proposed semi-discrete DG methods, we further apply in time the implicit version of the spectral deferred correction (SDC) methods [22]. Such methods can be easily constructed to have arbitrary order of accuracy, and they only need to store the numerical solution at the n-th time level in order to compute the solution at $(n+1)$-th time level. In [28], Xia et al demonstrated that the implicit SDC methods provide efficient time discretizations for the LDG methods to solve PDEs with high order spatial derivatives.

The remaining of this paper is organized as follows. In Section 2, notations are introduced for meshes, discrete spaces and projection operators. In Sections 3-5, we propose and analyze DG methods for even-order equations (1.1)-(1.3), the third order wave equation (1.4), and the linear Schrödinger equation (1.5), respectively. The presentation in each section starts with the method, energy relations for numerical stability, and error estimates. Parameters in the numerical fluxes are identified for the L^2 stability and for the optimality of the accuracy of the proposed methods. In Section 6, numerical examples are presented to verify our theoretical results. The concluding remarks are given in Section 7.

2 Discrete spaces and projections

Let the computational domain be $\Omega = [x_{min}, x_{max}]$, with a partition or mesh $x_{min} = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = x_{max}$. Let $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ denote an element with the length $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, and $h = \max_{1\le j\le N} \Delta x_j$. Define $\mathcal{T}_h = \{I_j : j = 1, 2, \cdots, N\}$. The following discrete space will be used,

$$\mathcal{V}_h^k = \{v : v|_{I_j} \in P^k(I_j), \forall I_j \in \mathcal{T}_h\}, \tag{2.1}$$

where $P^k(I_j)$ is the space of polynomials with degree at most k in I_j. For any $v \in \mathcal{V}_h^k$, $v^+_{j+\frac{1}{2}}$ and $v^-_{j+\frac{1}{2}}$ denote the limit values of v at $x_{j+\frac{1}{2}}$ from the right element I_{j+1} and from the left element I_j, respectively. As usual, $\{v\}_{j+\frac{1}{2}} = \frac{1}{2}(v^+_{j+\frac{1}{2}} + v^-_{j+\frac{1}{2}})$ and $[v]_{j+\frac{1}{2}} = (v^+_{j+\frac{1}{2}} - v^-_{j+\frac{1}{2}})$ represent, respectively, the average and the jump of the function v at $x_{j+\frac{1}{2}}$ for any j. We also define

$$F_1(\varphi, \psi, \beta_1) = \{\varphi\} + \alpha[\varphi] + \beta_1[\psi], \tag{2.2}$$

$$F_2(\psi, \varphi, \beta_2) = \{\psi\} - \alpha[\psi] + \beta_2[\varphi], \tag{2.3}$$

at cell interfaces, where α, β_1, β_2 are constants that are $O(1)$ and they will be specified when being used, while $\psi, \varphi \in H^1(\mathcal{T}_h)$ are piecewise-defined with respect to the mesh and have well-defined left and right traces at mesh nodes. Note that both F_1 and F_2 involve the same parameter α, and we omit the α-dependence in notation for brevity.

For square integrable functions on a given domain K, the standard notations are used for the inner product and the L^2 norm, namely,

$$(v, w)_K := \int_K vw dx, \quad \|v\|_K := \sqrt{(v, v)_K}, \quad \forall v, w \in L^2(K). \tag{2.4}$$

When $K = \Omega$, we also write (v, w) and $\|v\|$.

Next, we will introduce the standard L^2 projection P_h, that projects a function $v \in L^2(\Omega)$ onto the discrete space $\mathcal{V}_h^k$, and the Gauss-Radau projections $P_h^{\pm}$, that project a function $v \in H^1(\mathcal{T}_h)$ onto $\mathcal{V}_h^k$. They are defined as follows,

$$\int_{I_j} (P_h v - v) w dx = 0, \quad \forall w \in P^k(I_j), \tag{2.5}$$

$$\int_{I_j} (P_h^+ v - v) w dx = 0, \quad \forall w \in P^{k-1}(I_j) \text{ and } P_h^+ v(x^+_{j-\frac{1}{2}}) = v(x^+_{j-\frac{1}{2}}), \tag{2.6}$$

$$\int_{I_j} (P_h^- v - v) w dx = 0, \quad \forall w \in P^{k-1}(I_j) \text{ and } P_h^- v(x^-_{j+\frac{1}{2}}) = v(x^-_{j+\frac{1}{2}}), \tag{2.7}$$

for any $j = 1, \cdots, N$, and have the following approximation property:

$$\|v - \pi_h v\|^2 + h \sum_j ((v - \pi_h v)^{\pm}_{j+\frac{1}{2}})^2 \leqslant C_* h^{2k+2} \|v\|^2_{H^{k+1}(\Omega)}, \quad \forall v \in H^{k+1}(\Omega), \tag{2.8}$$

where $\pi_h = P_h^{\pm}$ or P_h, and C_* is a positive constant depending on k but not on h or v. Throughout, the standard notations are used for the Sobolev space $H^{k+1}(\Omega)$ and its norm $\|\cdot\|_{H^{k+1}(\Omega)}$.

In our error estimates, we will frequently use the following linear operator that maps from $H^1(\Omega) \times H^1(\Omega)$ onto $\mathcal{V}_h^k \times \mathcal{V}_h^k$,

$$\Pi_h \begin{pmatrix} \varphi \\ \psi \end{pmatrix} := \begin{pmatrix} \Pi_h^1(\varphi, \psi, \beta_1) \\ \Pi_h^2(\psi, \varphi, \beta_2) \end{pmatrix} = \begin{pmatrix} P_h^+ \left((\frac{1}{2}+\alpha)\varphi + \beta_1\psi\right) + P_h^- \left((\frac{1}{2}-\alpha)\varphi - \beta_1\psi\right) \\ P_h^+ \left((\frac{1}{2}-\alpha)\psi + \beta_2\varphi\right) + P_h^- \left((\frac{1}{2}+\alpha)\psi - \beta_2\varphi\right) \end{pmatrix}. \tag{2.9}$$

One would want to keep in mind that the operator Π_h should have been written as $\Pi_h^{\alpha,\beta_1,\beta_2}$ and we omit the parameter dependence for brevity. Associated with the operator Π_h, we define $(\eta_\varphi, \eta_\psi)^{\mathrm{T}}$ as

$$\begin{pmatrix} \eta_\varphi \\ \eta_\psi \end{pmatrix} := \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \Pi_h \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi - \Pi_h^1(\varphi, \psi, \beta_1) \\ \psi - \Pi_h^2(\psi, \varphi, \beta_2) \end{pmatrix}. \tag{2.10}$$

Let $\varphi_h, \psi_h \in \mathcal{V}_h^k$ be some approximations for φ and ψ, respectively, we will also use the following notation in our analysis

$$\begin{pmatrix} \zeta_\varphi \\ \zeta_\psi \end{pmatrix} := \Pi_h \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} = \begin{pmatrix} \Pi_h^1(\varphi, \psi, \beta_1) - \varphi_h \\ \Pi_h^2(\psi, \varphi, \beta_2) - \psi_h \end{pmatrix}. \tag{2.11}$$

Note that the following decomposition of the errors $e_\varphi = \varphi - \varphi_h$ and $e_\psi = \psi - \psi_h$ holds

$$e_\varphi = \eta_\varphi + \zeta_\varphi, \quad e_\psi = \eta_\psi + \zeta_\psi. \tag{2.12}$$

The operator Π_h was first introduced in [9], and its main properties are summarized in the following lemma.

Lemma 2.1. (Lemma 2.4 in [9]). Consider $(\varphi, \psi) \in H^{k+1}(\Omega) \times H^{k+1}(\Omega)$. For any given α, β_1, β_2, the operator Π_h has the following properties:

$$(i) \quad \int_{I_j} \eta_\varphi v_x dx = 0, \quad \int_{I_j} \eta_\psi w_x dx = 0, \quad \forall v, w \in \mathcal{V}_h^k, \quad \forall j, \tag{2.13}$$

$$(ii) \quad \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \Pi_h \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\| \le C_* \left(1 + |\alpha| + \max(|\beta_1|, |\beta_2|)\right) h^{k+1} \left(\|\psi\|_{H^{k+1}(\Omega)} + \|\varphi\|_{H^{k+1}(\Omega)} \right). \tag{2.14}$$

If we further assume $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$, we have

$$(iii) \quad \Pi_h \text{ defines a projection, that is } (\Pi_h)^2 = \Pi_h, \tag{2.15}$$

$$(iv) \quad F_1(\eta_\varphi, \eta_\psi, \beta_1)_{j-\frac{1}{2}} = 0, \quad F_2(\eta_\psi, \eta_\varphi, \beta_2)_{j-\frac{1}{2}} = 0, \quad \forall j. \tag{2.16}$$

3 DG methods for even order equations

In this section, we consider even order equations, which include the heat equation (1.1), a fourth order equation (1.2), and the arbitrary even order equations (1.3). The boundary conditions are periodic. For each equation, we will start with its first order form, and apply a general DG formulation. The numerical fluxes are given as the linear combinations of averages values of fluxes, jumps of the solution and the auxiliary variables at cell interfaces, and they involve several parameters. We then identify the conditions on these parameters, such that the DG methods will be L^2 stable; the parameters are further specified in order for the solution and some of auxiliary variables to be optimally accurate in the L^2 norm. To prove the optimality of the schemes analytically, one or more than one projection operator will be designed, tailored for each choice of the numerical fluxes, in order to eliminate those terms at cell interfaces that would otherwise contribute to the sub-optimality of the error estimates. We want to point out that Section 4 and Section 5 follow a similar structure in the presentation.

3.1 DG methods for the heat equation

In this subsection, we will formulate and analyze DG methods for the heat equation (1.1). Start with the first order form of the equation,

$$u_t - p_x = 0, \quad p - u_x = 0. \tag{3.1}$$

A general DG method can be given as follows. Look for u_h, $p_h \in \mathcal{V}_h^k$ such that for any v, $w \in \mathcal{V}_h^k$, and for any j,

$$((u_h)_t, v)_{I_j} + (p_h, v_x)_{I_j} - \left(F_p v^-\right)_{j+\frac{1}{2}} + \left(F_p v^+\right)_{j-\frac{1}{2}} = 0, \tag{3.2}$$

$$(p_h, w)_{I_j} + (u_h, w_x)_{I_j} - \left(F_u w^-\right)_{j+\frac{1}{2}} + \left(F_u w^+\right)_{j-\frac{1}{2}} = 0. \tag{3.3}$$

Here, F_p and F_u in (3.2)-(3.3) are numerical fluxes, which are single-valued functions defined on the cell interfaces and should be designed to ensure the numerical stability and accuracy of numerical solutions. We here consider a family of numerical fluxes, namely,

$$F_p = F_1(p_h, u_h, \beta_1), \quad F_u = F_2(u_h, p_h, \beta_2), \tag{3.4}$$

and they will correspondingly define a family of DG methods. Note that the numerical fluxes (3.4) include some commonly used ones, such as the central fluxes with $\alpha = \beta_1 = \beta_2 = 0$, and the alternating fluxes with $\alpha = \pm\frac{1}{2}$, $\beta_1 = \beta_2 = 0$. When $\beta_2 = 0$, the auxiliary variable p_h could be locally expressed in terms of u_h, and the DG methods (3.2)-(3.4) become LDG methods.

With the periodic boundary conditions, we sum up the scheme (3.2)-(3.3) over j and reach a more compact form of the scheme: look for $u_h, p_h \in \mathcal{V}_h^k$ such that

$$B(u_h, p_h; v, w) = 0, \quad \forall v, w \in \mathcal{V}_h^k, \tag{3.5}$$

where

$$\begin{aligned} B(u_h, p_h; v, w) = &\int_\Omega (u_h)_t v dx + \sum_j \left(\int_{I_j} p_h v_x dx + (F_p[v])_{j-\frac{1}{2}} \right) \\ &+ \int_\Omega p_h w dx + \sum_j \left(\int_{I_j} u_h w_x dx + (F_u[w])_{j-\frac{1}{2}} \right). \end{aligned} \tag{3.6}$$

3.1.1 L^2 stability

In this subsection, the L^2 stability is established for the semi-discrete DG method (3.2)-(3.3) with the general numerical fluxes (3.4). The conditions on the parameters α, β_1, β_2 in the numerical fluxes are identified to ensure the stability.

Theorem 3.1. *With $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0$, the semi-discrete DG scheme* (3.2)-(3.3) *(or* (3.5)*) with the numerical fluxes* (3.4) *satisfies*

$$\|u_h(T)\|_\Omega^2 + 2\int_0^T \|p_h\|_\Omega^2 dt \le \|u_h(0)\|_\Omega^2, \tag{3.7}$$

where T is the final time.

Proof. We start with introducing $H(\varphi, \psi) = \sum_j H_j(\varphi, \psi)$, where

$$\begin{aligned} H_j(\varphi, \psi) = &\int_{I_j} \psi \varphi_x dx - (F_1(\psi, \varphi, \beta_1)\varphi^-)_{j+\frac{1}{2}} + (F_1(\psi, \varphi, \beta_1)\varphi^+)_{j-\frac{1}{2}} \\ &+ \int_{I_j} \varphi \psi_x dx - (F_2(\varphi, \psi, \beta_2)\psi^-)_{j+\frac{1}{2}} + (F_2(\varphi, \psi, \beta_2)\psi^+)_{j-\frac{1}{2}}. \end{aligned} \tag{3.8}$$

Using the definitions of the fluxes F_1, F_2 in (2.2)-(2.3), we have

$$H(\varphi, \psi) = \sum_j (-[\varphi\psi] + F_1[\varphi] + F_2[\psi])_{j-\frac{1}{2}} = \sum_j \left(\beta_1[\varphi]^2 + \beta_2[\psi]^2\right)_{j-\frac{1}{2}}. \tag{3.9}$$

We now take the test functions $v = u_h$, $w = p_h$ in (3.5), and with the definition of numerical fluxes F_p, F_u in (3.4), we get

$$B(u_h, p_h; u_h, p_h) = \frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx + \int_\Omega p_h^2 dx + H(u_h, p_h)$$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx + \int_\Omega p_h^2 dx + \sum_j \left(\beta_1[u_h]^2 + \beta_2[p_h]^2\right)_{j-\frac{1}{2}} = 0. \quad (3.10)$$

Finally under the conditions $\beta_1 \geqslant 0$ and $\beta_2 \geqslant 0$, we reach the energy stability relation,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx + \int_\Omega p_h^2 dx = -\sum_j \left(\beta_1[u_h]^2 + \beta_2[p_h]^2\right)_{j-\frac{1}{2}} \leq 0. \quad (3.11)$$

Integrating (3.11) over $[0, T]$, we get (3.7) about stability for equation (1.1). □

3.1.2 L^2 error estimates

In this subsection, we will establish that the DG methods (3.2)-(3.3) with a subfamily of the numerical fluxes (3.4) are optimally accurate in the L^2 norm when the exact solution is sufficiently smooth. The analysis is based on the energy relation in Theorem 3.1, approximation properties of the discrete space $\mathcal{V}_h^k$, and a special choice of a projection operator.

Theorem 3.2. *For the semi-discrete DG scheme* (3.2)-(3.3) *with the numerical fluxes* (3.4) *where the parameters satisfy* $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$ *and* $\beta_i \geq 0$, $i = 1, 2$, *the following error estimates hold when the exact solution* u *is sufficiently smooth,*

$$\|u - u_h\|_\Omega^2 \leq Ch^{2k+2}, \quad \int_0^T \|p - p_h\|_\Omega^2 dt \leq Ch^{2k+2}. \quad (3.12)$$

Here $p = u_x$, *and the constant* C *depends on* k, *the final time* T, $\|u\|_{L^\infty((0,T);H^{k+2}(\Omega))}$ *and* $\|u_t\|_{L^\infty((0,T);H^{k+1}(\Omega))}$ *but not on* h.

Proof. Since the numerical fluxes (3.4) are consistent, the exact solution u of the heat equation (1.1) and $p = u_x$ satisfy

$$B(u, p; v, w) = 0, \quad \forall v, w \in \mathcal{V}_h^k, \quad (3.13)$$

hence we get the error equation

$$B(e_u, e_p; v, w) = 0, \quad \forall v, w \in \mathcal{V}_h^k, \quad (3.14)$$

where $e_p = p - p_h$ and $e_u = u - u_h$. By using the following projection,

$$\Pi_h \begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} \Pi_h^1(p, u, \beta_1) \\ \Pi_h^2(u, p, \beta_2) \end{pmatrix}, \quad (3.15)$$

we can decompose the errors e_p and e_u into $e_p = \eta_p + \zeta_p$ and $e_u = \eta_u + \zeta_u$ based on (2.10)-(2.11). With the linearity of B, the error equation (3.14) becomes

$$B(\zeta_u, \zeta_p; v, w) = -B(\eta_u, \eta_p; v, w), \quad \forall v, w \in \mathcal{V}_h^k. \quad (3.16)$$

We now take $v=\zeta_u$, $w=\zeta_p$ in (3.16). Following the similar derivation to get (3.10) and the definition of B in (3.6), we have

$$B(\zeta_u,\zeta_p;\zeta_u,\zeta_p) = \frac{1}{2}\frac{d}{dt}\int_\Omega \zeta_u^2 dx + \int_\Omega \zeta_p^2 dx + \sum_j \left(\beta_1[\zeta_u]^2 + \beta_2[\zeta_p]^2\right)_{j-\frac{1}{2}}, \tag{3.17}$$

$$\begin{aligned} B(\eta_u,\eta_p;\zeta_u,\zeta_p) = &\int_\Omega (\eta_u)_t \zeta_u dx + \int_\Omega \eta_p\zeta_p dx + \sum_j \int_{I_j} (\eta_p(\zeta_u)_x + \eta_u(\zeta_p)_x) dx \\ &+ \sum_j \left(F_1(\eta_p,\eta_u,\beta_1)[\zeta_u] + F_2(\eta_u,\eta_p,\beta_2)[\zeta_p]\right)_{j-\frac{1}{2}}. \end{aligned} \tag{3.18}$$

Under the assumption $\alpha^2+\beta_1\beta_2=\frac{1}{4}$, we can use the properties of Π_h in Lemma 2.1 and get

$$\int_{I_j} \eta_p(\zeta_u)_x dx = 0, \int_{I_j} \eta_u(\zeta_p)_x dx = 0, \quad F_1(\eta_p,\eta_u,\beta_1)=0, F_2(\eta_u,\eta_p,\beta_2)=0. \tag{3.19}$$

Combining (3.16)-(3.19), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \zeta_u^2 dx + \int_\Omega \zeta_p^2 dx + \sum_j \left(\beta_1[\zeta_u]^2 + \beta_2[\zeta_p]^2\right)_{j-\frac{1}{2}} &= -\int_\Omega (\eta_u)_t\zeta_u dx - \int_\Omega \eta_p\zeta_p dx \\ \le \frac{1}{2}\int_\Omega (2(\eta_u)_t^2 + (\eta_p)^2) dx + \frac{1}{4}\int_\Omega (\zeta_u^2 + 2\zeta_p^2) dx. \end{aligned} \tag{3.20}$$

Based on the approximation property (2.14) of Π_h and $(\eta_u)_t = \eta_{u_t}$, we know $\|(\eta_u)_t\| \le Ch^{k+1}$ and $\|\eta_p\| \le Ch^{k+1}$. Therefore,

$$\frac{d}{dt}\int_\Omega \zeta_u^2 dx + \int_\Omega \zeta_p^2 dx + 2\sum_j \left(\beta_1[\zeta_u]^2 + \beta_2[\zeta_p]^2\right)_{j-\frac{1}{2}} \le Ch^{2k+2} + \frac{1}{2}\int_\Omega \zeta_u^2 dx. \tag{3.21}$$

Note that we use the initialization $u_h(x,0) = P_h u(x,0)$ which can be bounded by

$$\left\|\Pi_h^2(u,p,\beta_2) - u_h\right\|_{t=0} = \left\|\Pi_h^2(u,p,\beta_2) - u + u - P_h u\right\|_{t=0} \le Ch^{k+1}. \tag{3.22}$$

Now we can apply the Gronwall's inequality and obtain

$$\|\zeta_u(\cdot,t)\|_\Omega^2 \le Ch^{2k+2}, \quad \int_0^T \|\zeta_p\|_\Omega^2 dt \le Ch^{2k+2}.$$

Finally, using the approximation results in (2.14), we reach

$$\begin{aligned} \|u-u_h\|_\Omega^2 \le& 2(\|\zeta_u\|_\Omega^2 + \|\eta_u\|_\Omega^2) \le Ch^{2k+2}, \\ \int_0^T \|p-p_h\|_\Omega^2 dt \le& \int_0^T 2(\|\zeta_p\|_\Omega^2 + \|\eta_p\|_\Omega^2) dt \le Ch^{2k+2}. \end{aligned}$$

Through the proof, the constant C depends on k, the finial time T, $\|u\|_{L^\infty((0,T);H^{k+2}(\Omega))}$ and $\|u_t\|_{L^\infty((0,T);H^{k+1}(\Omega))}$ but not on h. □

Remark 3.3. Similar as in [9], the DG methods with the more general fluxes

$$F_p = \{p_h\} + \alpha_1[p_h] + \beta_1[u_h], \quad F_u = \{u_h\} + \alpha_2[u_h] + \beta_2[p_h],$$

where $\beta_i \geq 0, i = 1, 2$ and $(\alpha_1 + \alpha_2)^2 \leq 4\beta_1\beta_2$, also have the energy stability (3.7). Such DG methods however are often sub-optimal in their accuracy.

Remark 3.4. Just as in Theorem 2.6 of [9], the condition $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$ on the parameters in the numerical fluxes can be further relaxed in order for the proposed schemes to be L^2 optimal when solving the heat equation. For instance, we can require

$$\alpha^2 + \beta_1\beta_2 = \frac{1}{4} + ch^\delta, \quad \delta \geq \frac{1}{2}, \quad \min(\beta_1, \beta_2) > 0,$$

where c is a constant independent of h.

3.2 DG methods for a fourth order equation

In this subsection, we will formulate and analyze DG methods for the fourth order equation (1.2). We start with rewriting the equation into its first order system,

$$u_t + p_x = 0, \quad p = q_x, \quad q = r_x, \quad r = u_x, \tag{3.23}$$

and then apply a general DG method. That is, to look for $u_h, p_h, q_h, r_h \in \mathcal{V}_h^k$ such that for any $v, w, z, g \in \mathcal{V}_h^k$ and for any j

$$((u_h)_t, v)_{I_j} - (p_h, v_x)_{I_j} + \left(F_p v^-\right)_{j+\frac{1}{2}} - \left(F_p v^+\right)_{j-\frac{1}{2}} = 0, \tag{3.24}$$

$$(p_h, w)_{I_j} + (q_h, w_x)_{I_j} - \left(F_q w^-\right)_{j+\frac{1}{2}} + \left(F_q w^+\right)_{j-\frac{1}{2}} = 0, \tag{3.25}$$

$$(q_h, z)_{I_j} + (r_h, z_x)_{I_j} - \left(F_r z^-\right)_{j+\frac{1}{2}} + \left(F_r z^+\right)_{j-\frac{1}{2}} = 0, \tag{3.26}$$

$$(r_h, g)_{I_j} + (u_h, g_x)_{I_j} - \left(F_u g^-\right)_{j+\frac{1}{2}} + \left(F_u g^+\right)_{j-\frac{1}{2}} = 0. \tag{3.27}$$

The terms F_p, F_q, F_r and F_u are numerical fluxes. They are defined as linear combinations of the averages of fluxes and the jumps of the unknown solutions u_h and p_h, q_h and r_h, and are chosen as

$$F_p = F_1(p_h, u_h, -\beta_1),\ F_q = F_2(q_h, r_h, \beta_2),\ F_r = F_1(r_h, q_h, \beta_1),\ F_u = F_2(u_h, p_h, -\beta_2). \tag{3.28}$$

In order for simplifying the conditions and flexibly extending to general even-order equations, the parameter α is the same for all F_p, F_u, F_r, F_q, while β_1, β_2 in F_p, F_u are related to those in F_r, F_q.

With the periodic boundary conditions, we sum up all the equations in (3.24) -(3.27) over j and obtain a more compact form of the scheme: look for $u_h, p_h, q_h, r_h \in \mathcal{V}_h^k$ such that

$$B(u_h, p_h, q_h, r_h; v, w, z, g) = 0, \quad \forall v, w, z, g \in \mathcal{V}_h^k, \tag{3.29}$$

where

$$\begin{aligned} B(u_h, p_h, q_h, r_h; v, w, z, g) = &\int_\Omega ((u_h)_t v + p_h w + q_h z + r_h g)\, dx \\ &+ \sum_j \int_{I_j} (-p_h v_x + q_h w_x + r_h z_x + u_h g_x)\, dx \\ &+ \sum_j (-F_p[v] + F_q[w] + F_r[z] + F_u[g])_{j-\frac{1}{2}}. \end{aligned} \tag{3.30}$$

3.2.1 L^2 stability

In this subsection, the L^2 stability is established for the semi-discrete DG method (3.24)-(3.27) with the general numerical fluxes (3.28). Just as in [20], in order for the proposed methods to be optimally accurate, more than one energy equation is needed in the presence of multiple auxiliary unknowns, also see Remark 3.6.

• **The first energy equation.** By taking the test functions $v = u_h$, $w = r_h$, $z = q_h$ and $g = -p_h$ in (3.29), we obtain

$$\begin{aligned} 0 =& B(u_h, p_h, q_h, r_h; u_h, r_h, q_h, -p_h) = \int_\Omega \left((u_h)_t u_h + q_h^2\right) dx \\ &+ \sum_j ([u_h p_h] - [q_h r_h] - F_p[u_h] + F_q[r_h] + F_r[q_h] - F_u[p_h])_{j-\frac{1}{2}}. \end{aligned} \tag{3.31}$$

Using the definitions of F_p, F_q, F_r, F_u in (3.28), we get

$$\begin{aligned} 0 =& B(u_h, p_h, q_h, r_h; u_h, r_h, q_h, -p_h) = \int_\Omega \left((u_h)_t u_h + q_h^2\right) dx \\ &+ \sum_j \left(\beta_1([u_h]^2 + [q_h]^2) + \beta_2([p_h]^2 + [r_h]^2)\right)_{j-\frac{1}{2}}. \end{aligned} \tag{3.32}$$

• **The second energy equation.** We next take the time derivative for the equation (3.27) and sum it up, together with (3.24)-(3.26), over j. By taking the test functions $v = -\frac{1}{2}q_h$, $w = \frac{1}{2}p_h$, $z = \frac{1}{2}(u_h)_t$ and $g = \frac{1}{2}r_h$, we get

$$0 = B(u_h, p_h, q_h, (r_h)_t; -\frac{1}{2}q_h, \frac{1}{2}p_h, \frac{1}{2}(u_h)_t, \frac{1}{2}r_h) = \frac{1}{2}\int_\Omega \left(p_h^2 + (r_h)_t r_h\right) dx$$

$$+\frac{1}{2}\sum_j \int_{I_j} (p_h(q_h)_x + q_h(p_h)_x + r_h(u_h)_{tx} + (u_h)_t(r_h)_x)\, dx$$
$$+\frac{1}{2}\sum_j (F_p[q_h] + F_q[p_h] + F_r[(u_h)_t] + (F_u)_t[r_h])_{j-\frac{1}{2}}\,. \tag{3.33}$$

Using the definitions of F_p, F_q, F_r, F_u in (3.28), we have

$$0 = B(u_h, p_h, q_h, (r_h)_t; -\frac{1}{2}q_h, \frac{1}{2}p_h, \frac{1}{2}(u_h)_t, \frac{1}{2}r_h) = \frac{1}{2}\int_\Omega \left(p_h^2 + (r_h)_t r_h\right) dx$$
$$+\frac{1}{2}\sum_j (\beta_1([q_h][(u_h)_t] - [u_h][q_h]) + \beta_2([r_h][p_h] - [(p_h)_t][r_h]))_{j-\frac{1}{2}}\,. \tag{3.34}$$

• **The third energy equation.** Here, we take the time derivative for the equation (3.26)-(3.27) and sum it up, together with (3.24)-(3.25), over j. Then, we take the test functions $v = \frac{1}{2}(u_h)_t$, $w = \frac{1}{2}(r_h)_t$, $z = \frac{1}{2}q_h$ and $g = -\frac{1}{2}p_h$. Using the definitions of (3.29) and F_p, F_q, F_r, F_u in (3.28), we have

$$0 = B(u_h, p_h, (q_h)_t, (r_h)_t; \frac{1}{2}(u_h)_t, \frac{1}{2}(r_h)_t, \frac{1}{2}q_h, -\frac{1}{2}p_h) = \frac{1}{2}\int_\Omega \left((u_h)_t^2 + q_h(q_h)_t\right) dx$$
$$+\frac{1}{2}\sum_j ([(u_h)_t p_h] - [q_h(r_h)_t] - F_p[(u_h)_t] + F_q[(r_h)_t] + (F_r)_t[q_h] - (F_u)_t[p_h])_{j-\frac{1}{2}}$$
$$= \frac{1}{2}\int_\Omega \left((u_h)_t^2 + (q_h)_t q_h\right) dx$$
$$+\frac{1}{2}\sum_j (\beta_1([(q_h)_t][q_h] + [u_h][(u_h)_t]) + \beta_2([r_h][(r_h)_t] + [(p_h)_t][p_h]))_{j-\frac{1}{2}}. \tag{3.35}$$

• **The fourth energy equation.** We take the time derivative for (3.25)-(3.27) and sum them up with (3.24) over j. Using the definition of B in (3.29) and taking the test functions $v = -\frac{1}{2}(q_h)_t$, $w = \frac{1}{2}p_h$, $z = \frac{1}{2}(u_h)_t$ and $g = \frac{1}{2}(r_h)_t$, we obtain

$$0 = B(u_h, (p_h)_t, (q_h)_t, (r_h)_t; -\frac{1}{2}(q_h)_t, \frac{1}{2}p_h, \frac{1}{2}(u_h)_t, \frac{1}{2}(r_h)_t) = \frac{1}{2}\int_\Omega \left((p_h)_t p_h + ((r_h)_t)^2\right) dx$$
$$+\frac{1}{2}\sum_j (-[(u_h)_t(r_h)_t] - [p_h(q_h)_t] + F_p[(q_h)_t] + (F_q)_t[p_h] + (F_r)_t[(u_h)_t] + (F_u)_t[(r_h)_t])_{j-\frac{1}{2}}$$
$$= \frac{1}{2}\int_\Omega \left((p_h)_t p_h + ((r_h)_t)^2\right) dx$$
$$+\frac{1}{2}\sum_j (\beta_1([(q_h)_t][(u_h)_t] - [(q_h)_t][u_h]) + \beta_2([(r_h)_t][p_h] - [(p_h)_t][(r_h)_t])_{j-\frac{1}{2}}. \tag{3.36}$$

• **The fifth energy equation.** Finally we take the time derivative for equations (3.24)-(3.27), sum them up over j. With the test functions $v = (u_h)_t$, $w = (r_h)_t$,

$z = (q_h)_t$ and $g = -(p_h)_t$, we get

$$0 = B((u_h)_t, (p_h)_t, (q_h)_t, (r_h)_t; (u_h)_t, (r_h)_t, (q_h)_t, -(p_h)_t) = \int_\Omega \left((u_h)_{tt}(u_h)_t + (q_h)_t^2\right) dx$$
$$+ \sum_j \left(\beta_1([(u_h)_t]^2 + [(q_h)_t]^2) + \beta_2([(p_h)_t]^2 + [(r_h)_t]^2)\right)_{j-\frac{1}{2}}. \qquad (3.37)$$

Combining (3.32) and (3.34)-(3.37), we have

$$\frac{1}{4}\frac{d}{dt}\int_\Omega \left(2u_h^2 + p_h^2 + q_h^2 + r_h^2 + 2(u_h)_t^2\right) dx + \frac{1}{2}\int_\Omega \left(2q_h^2 + p_h^2 + (u_h)_t^2 + (r_h)_t^2 + 2(q_h)_t^2\right) dx$$
$$+ \sum_j \Big(\beta_1 \Lambda([u_h], [(u_h)_t], [q_h], [(q_h)_t]) + \beta_2 \Lambda([(p_h)_t], [p_h], [r_h], [(r_h)_t])\Big)_{j-\frac{1}{2}} = 0, \quad (3.38)$$

where $\Lambda(a, b, c, d)$ is a non-negative quadratic form, defined as

$$\Lambda(a, b, c, d) = a^2 + b^2 + c^2 + d^2 + \frac{1}{2}(ab + cd + bc - ac + bd - ad) \geq 0. \qquad (3.39)$$

If we require $\beta_1 \geq 0, \beta_2 \geq 0$, then (3.38) gives

$$\frac{1}{4}\frac{d}{dt}\int_\Omega \left(2u_h^2 + p_h^2 + q_h^2 + r_h^2 + 2(u_h)_t^2\right) dx + \int_\Omega \left(q_h^2 + \frac{1}{2}p_h^2 + r_h^2 + \frac{1}{2}(r_h)_t^2 + (q_h)_t^2\right) dx \leq 0. \qquad (3.40)$$

This leads to the following theorem.

Theorem 3.5. *Using the numerical fluxes* (3.28) *with* $\beta_1 \geq 0$ *and* $\beta_2 \geq 0$*, the numerical solutions of the semi-discrete DG method* (3.29) *for the fourth order equation satisfy*

$$2\|u_h(T)\|_\Omega^2 + \|p_h(T)\|_\Omega^2 + \|q_h(T)\|_\Omega^2 + \|r_h(T)\|_\Omega^2 + 2\|(u_h)_t(T)\|_\Omega^2$$
$$\leq 2\|u_h(0)\|_\Omega^2 + \|p_h(0)\|_\Omega^2 + \|q_h(0)\|_\Omega^2 + \|r_h(0)\|_\Omega^2 + 2\|(u_h)_t(0)\|_\Omega^2.$$

Here T *is the final time.*

Remark 3.6. Compared with the analysis in [20], more energy equations are needed in our analysis to ensure the optimal accuracy of the proposed methods due to the extra parameters β_1, β_2 and α ($\alpha \neq \pm\frac{1}{2}$). When $\beta_1 = \beta_2 = 0$, only the first and the second energy equations are needed just as in [20].

3.2.2 L^2 error estimates

In this subsection, we will prove the optimal *a priori* L^2 error estimate for the DG method (3.29) for the fourth order equation when the exact solution is sufficiently smooth, under the following assumption for the parameters in the numerical fluxes (3.28)

$$\alpha^2 + \beta_1\beta_2 = \frac{1}{4}, \quad \beta_1 \geq 0,\ \beta_2 \geq 0. \tag{3.41}$$

Since the numerical fluxes (3.28) are consistent, the exact solution u, $r = u_x$, $q = r_x$, and $p = q_x$ satisfy

$$B(u, p, q, r; v, w, z, g) = 0, \quad \forall v, w, z, g \in \mathcal{V}_h^k, \tag{3.42}$$

hence we get the error equation

$$B(e_u, e_p, e_q, e_r; v, w, z, g) = 0, \quad \forall v, w, z, g \in \mathcal{V}_h^k. \tag{3.43}$$

Here $e_\phi = \phi - \phi_h$, with $\phi = u, p, q, r$ are error functions. To ensure the error estimates to be optimal, we use the special projections

$$\Pi_h \begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} \Pi_h^1(p, u, -\beta_1) \\ \Pi_h^2(u, p, -\beta_2) \end{pmatrix}, \quad \Pi_h \begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} \Pi_h^1(r, q, \beta_1) \\ \Pi_h^2(q, r, \beta_2) \end{pmatrix}, \tag{3.44}$$

with which the error functions can be decomposed into $e_\phi = \eta_\phi + \zeta_\phi, \phi = u, p, q, r$ based on (2.10)-(2.11), and the error equation becomes

$$B(\zeta_u, \zeta_p, \zeta_q, \zeta_r; v, w, z, g) = -B(\eta_u, \eta_p, \eta_q, \eta_r; v, w, z, g), \quad \forall v, w, z, g \in \mathcal{V}_h^k. \tag{3.45}$$

We choose to set the initial condition $p_h(x, 0)$ as follows,

$$p_h(x, 0) = P_h^+ p(x, 0), \quad p(x, 0) = u_{xxx}(x, 0). \tag{3.46}$$

Using $p_h(x, 0)$, we can further define the initial data $q_h(x, 0), r_h(x, 0), u_h(x, 0) \in \mathcal{V}_h^k$, satisfying

$$\begin{aligned} &(p_h, w)_{I_j} + (q_h, w_x)|_{I_j} - (\widehat{q}_h w^-)|_{j+\frac{1}{2}} + (\widehat{q}_h w^+)|_{j-\frac{1}{2}} = 0,\ \forall w \in \mathcal{V}_h^k, \\ &\qquad\qquad \text{with } \widehat{q}_h = q_h^-(x, 0) \text{ and } \int_\Omega q_h(x, 0)dx = \int_\Omega q(x, 0)dx, \end{aligned} \tag{3.47}$$

$$\begin{aligned} &(q_h, w)_{I_j} + (r_h, w_x)|_{I_j} - (\widehat{r}_h w^-)|_{j+\frac{1}{2}} + (\widehat{r}_h w^+)|_{j-\frac{1}{2}} = 0,\ \forall w \in \mathcal{V}_h^k, \\ &\qquad\qquad \text{with } \widehat{r}_h = r_h^+(x, 0) \text{ and } \int_\Omega r_h(x, 0)dx = \int_\Omega r(x, 0)dx, \end{aligned} \tag{3.48}$$

$$\begin{aligned} &(r_h, w)_{I_j} + (u_h, w_x)|_{I_j} - (\widehat{u}_h w^-)|_{j+\frac{1}{2}} + (\widehat{u}_h w^+)|_{j-\frac{1}{2}} = 0,\ \forall w \in \mathcal{V}_h^k, \\ &\qquad\qquad \text{with } \widehat{u}_h = u_h^-(x, 0) \text{ and } \int_\Omega u_h(x, 0)dx = \int_\Omega u(x, 0)dx. \end{aligned} \tag{3.49}$$

Following a similar analysis for Lemma 5.1 in [23], we can prove that the initial conditions above are well-defined with optimal accuracy. And similar to the analysis for Lemma 2.4 in [30], we have the optimal error estimates about $||u_t - (u_h)_t||$ at $t = 0$. The results are summarized next with the proof omitted.

Lemma 3.7. Assuming $u(x,0)$ is sufficiently smooth, the initial conditions in (3.46)-(3.49) are well defined and satisfy the following estimates

$$\begin{aligned}
&||p(x,0)-p_h(x,0)||_\Omega \le Ch^{k+1},\ ||q(x,0)-q_h(x,0)||_\Omega \le Ch^{k+1},\\
&||u(x,0)-u_h(x,0)||_\Omega \le Ch^{k+1},\ ||r(x,0)-r_h(x,0)||_\Omega \le Ch^{k+1},\\
&||u_t(x,0)-(u_h)_t(x,0)||_\Omega \le Ch^{k+1}.
\end{aligned}$$

Here $(u_h)_t(x,0)$ is determined by (3.24) with $F_p = p_h^+(x,0)$ at $t=0$. And C depends on $||u(x,0)||_{H^{k+3}(\Omega)}$, and $r(x,0)=u_x(x,0), q(x,0)=u_{xx}(x,0)$.

Next, we will follow the analysis of energy stability and get five error equations to obtain the error estimates.

• **The first error equation** We start with the error equation (3.45) and take the test functions to be $v=\zeta_u, w=\zeta_r, z=\zeta_q, g=-\zeta_p$, and get

$$B(\zeta_u,\zeta_p,\zeta_q,\zeta_r;\zeta_u,\zeta_r,\zeta_q,-\zeta_p) = -B(\eta_u,\eta_p,\eta_q,\eta_r;\zeta_u,\zeta_r,\zeta_q,-\zeta_p). \tag{3.50}$$

Following the derivation of the first energy equation (3.32), we have

$$\begin{aligned}
B(\zeta_u,\zeta_p,\zeta_q,\zeta_r;\zeta_u,\zeta_r,\zeta_q,-\zeta_p) = &\int_\Omega \left((\zeta_u)_t\zeta_u + \zeta_q^2\right) dx\\
&+\sum_j \left(\beta_1([\zeta_u]^2+[\zeta_q]^2)+\beta_2([\zeta_p]^2+[\zeta_r]^2)\right)_{j-\frac{1}{2}}.
\end{aligned} \tag{3.51}$$

And from (3.30),

$$\begin{aligned}
&B(\eta_u,\eta_p,\eta_q,\eta_r;\zeta_u,\zeta_r,\zeta_q,-\zeta_p) = \int_\Omega ((\eta_u)_t\zeta_u+\eta_p\zeta_r+\eta_q\zeta_q-\eta_r\zeta_p)\,dx\\
&+\sum_j\int_{I_j}((-\eta_p)(\zeta_u)_x+\eta_q(\zeta_r)_x+\eta_r(\zeta_q)_x-\eta_u(\zeta_p)_x)\,dx\\
&+\sum_j(-F_p[\zeta_u]+F_q[\zeta_r]+F_r[\zeta_q]-F_u[\zeta_p])_{j-\frac{1}{2}}\\
&=\int_\Omega((\eta_u)_t\zeta_u+\eta_p\zeta_r+\eta_q\zeta_q-\eta_r\zeta_p)\,dx.
\end{aligned} \tag{3.52}$$

Here, the properties (2.13) and (2.16) of Π_h in Lemma 2.1 are used. Combining (3.50)-(3.52), we get

$$\begin{aligned}
&\int_\Omega \left((\zeta_u)_t\zeta_u+\zeta_q^2\right)dx + \sum_j\left(\beta_1([\zeta_u]^2+[\zeta_q]^2)+\beta_2([\zeta_p]^2+[\zeta_r]^2)\right)_{j-\frac{1}{2}}\\
&=-\int_\Omega((\eta_u)_t\zeta_u+\eta_p\zeta_r+\eta_q\zeta_q-\eta_r\zeta_p)\,dx.
\end{aligned} \tag{3.53}$$

• **The second error equation** Following a similar procedure to get the energy equation (3.34), we have

$$B(\zeta_u,\zeta_p,\zeta_q,(\zeta_r)_t;-\frac{1}{2}\zeta_q,\frac{1}{2}\zeta_p,\frac{1}{2}(\zeta_u)_t,\frac{1}{2}\zeta_r)=-B(\eta_u,\eta_p,\eta_q,(\eta_r)_t;-\frac{1}{2}\zeta_q,\frac{1}{2}\zeta_p,\frac{1}{2}(\zeta_u)_t,\frac{1}{2}\zeta_r),$$

and

$$\begin{aligned}B(\zeta_u,\zeta_p,\zeta_q,(\zeta_r)_t;-\frac{1}{2}\zeta_q,\frac{1}{2}\zeta_p,\frac{1}{2}(\zeta_u)_t,\frac{1}{2}\zeta_r)&=\frac{1}{2}\int_\Omega\left(\zeta_p^2+(\zeta_r)_t\zeta_r\right)dx\\+\frac{1}{2}\sum_j\left(\beta_1([\zeta_q][(\zeta_u)_t]-[\zeta_u][\zeta_q])+\beta_2([\zeta_r][\zeta_p]-[(\zeta_p)_t][\zeta_r])\right)_{j-\frac{1}{2}}.\end{aligned}\tag{3.54}$$

Using the definition of B in (3.30) and the properties (2.13), (2.16) of Π_h, we get

$$\begin{aligned}&B(\eta_u,\eta_p,\eta_q,(\eta_r)_t;-\frac{1}{2}\zeta_q,\frac{1}{2}\zeta_p,\frac{1}{2}(\zeta_u)_t,\frac{1}{2}\zeta_r)\\=&\frac{1}{2}\int_\Omega(-(\eta_u)_t\zeta_q+\eta_p\zeta_p+\eta_q(\zeta_u)_t+(\eta_r)_t\zeta_r)\,dx,\end{aligned}\tag{3.55}$$

hence

$$\begin{aligned}&\frac{1}{2}\int_\Omega\left(\zeta_p^2+(\zeta_r)_t\zeta_r\right)dx+\frac{1}{2}\sum_j\left(\beta_1([\zeta_q][(\zeta_u)_t]-[\zeta_u][\zeta_q])+\beta_2([\zeta_r][\zeta_p]-[(\zeta_p)_t][\zeta_r])\right)_{j-\frac{1}{2}}\\&=-\frac{1}{2}\int_\Omega(-(\eta_u)_t\zeta_q+\eta_p\zeta_p+\eta_q(\zeta_u)_t+(\eta_r)_t\zeta_r)\,dx.\end{aligned}\tag{3.56}$$

• **The third error equation.** Following a similar procedure to get the energy equation (3.35), we have

$$\begin{aligned}B(\zeta_u,\zeta_p,(\zeta_q)_t,(\zeta_r)_t;\frac{1}{2}(\zeta_u)_t,\frac{1}{2}(\zeta_r)_t,\frac{1}{2}\zeta_q,-\frac{1}{2}\zeta_p)&=\frac{1}{2}\int_\Omega\left((\zeta_u)_t^2+\zeta_q(\zeta_q)_t\right)dx\\+\frac{1}{2}\sum_j\left(\beta_1([(\zeta_q)_t][\zeta_q]+[\zeta_u][(\zeta_u)_t])+\beta_2([\zeta_r][(\zeta_r)_t]+[(\zeta_p)_t][\zeta_p])\right)_{j-\frac{1}{2}}.\end{aligned}\tag{3.57}$$

Using the definition of (3.30) and the properties (2.13), (2.16) of Π_h, we obtain

$$B(\eta_u,\eta_p,(\eta_q)_t,(\eta_r)_t;\frac{1}{2}(\zeta_u)_t,\frac{1}{2}(\zeta_r)_t,\frac{1}{2}\zeta_q,-\frac{1}{2}\zeta_p)=\frac{1}{2}\int_\Omega((\eta_u)_t(\zeta_u)_t+\eta_p(\zeta_r)_t+(\eta_q)_t\zeta_q-(\eta_r)_t\zeta_p)\,dx,$$

therefore

$$\begin{aligned}&\frac{1}{2}\int_\Omega\left((\zeta_q)_t\zeta_q+(\zeta_u)_t^2\right)dx+\frac{1}{2}\sum_j\left(\beta_1([(\zeta_q)_t][\zeta_q]+[\zeta_u][(\zeta_u)_t])+\beta_2([\zeta_r][(\zeta_r)_t]+[(\zeta_p)_t][\zeta_p])\right)_{j-\frac{1}{2}}\\&=-\frac{1}{2}\int_\Omega((\eta_u)_t(\zeta_u)_t+\eta_p(\zeta_r)_t+(\eta_q)_t\zeta_q-(\eta_r)_t\zeta_p)\,dx.\end{aligned}\tag{3.58}$$

• **The fourth error equation.** Here, we follow the procedure to derive (3.36) and have

$$B(\zeta_u,(\zeta_p)_t,(\zeta_q)_t,(\zeta_r)_t;-\frac{1}{2}(\zeta_q)_t,\frac{1}{2}\zeta_p,\frac{1}{2}(\zeta_u)_t,\frac{1}{2}(\zeta_r)_t)=\frac{1}{2}\int_\Omega\left((\zeta_p)_t\zeta_p+((\zeta_r)_t)^2\right)dx$$
$$+\frac{1}{2}\sum_j\left(\beta_1([(\zeta_q)_t][(\zeta_u)_t]-[\zeta_u][(\zeta_q)_t])+\beta_2([(\zeta_r)_t][\zeta_p]-[(\zeta_p)_t][(\zeta_r)_t])\right)_{j-\frac{1}{2}}. \tag{3.59}$$

And from (3.30) and the properties (2.13), (2.16) of Π_h, we get

$$B(\eta_u,(\eta_p)_t,(\eta_q)_t,(\eta_r)_t;-\frac{1}{2}(\zeta_q)_t,\frac{1}{2}\zeta_p,\frac{1}{2}(\zeta_u)_t,\frac{1}{2}(\zeta_r)_t)$$
$$=\frac{1}{2}\int_\Omega\left(-(\eta_u)_t(\zeta_q)_t+(\eta_p)_t\zeta_p+(\eta_q)_t(\zeta_u)_t+(\eta_r)_t(\zeta_r)_t\right)dx, \tag{3.60}$$

therefore

$$\frac{1}{2}\int_\Omega\left((\zeta_p)_t\zeta_p+((\zeta_r)_t)^2\right)dx+\frac{1}{2}\sum_j\left(\beta_1([(\zeta_q)_t][(\zeta_u)_t]-[\zeta_u][(\zeta_q)_t])+\beta_2([(\zeta_r)_t][\zeta_p]-[(\zeta_p)_t][(\zeta_r)_t])\right)_{j-\frac{1}{2}}$$
$$=-\frac{1}{2}\int_\Omega\left(-(\eta_u)_t(\zeta_q)_t+(\eta_p)_t\zeta_p+(\eta_q)_t(\zeta_u)_t+(\eta_r)_t(\zeta_r)_t\right)dx. \tag{3.61}$$

• **The fifth error equation.** Last, we follow the procedure to get the fifth energy equation (3.37) and have

$$B((\zeta_u)_t,(\zeta_p)_t,(\zeta_q)_t,(\zeta_r)_t;(\zeta_u)_t,(\zeta_r)_t,(\zeta_q)_t,-(\zeta_p)_t))=\int_\Omega\left((\zeta_u)_{tt}(\zeta_u)_t+(\zeta_q)_t^2\right)dx$$
$$+\sum_j\left(\beta_1([(\zeta_q)_t]^2+[(\zeta_u)_t]^2)+\beta_2([(\zeta_p)_t]^2+[(\zeta_r)_t]^2)\right)_{j-\frac{1}{2}}. \tag{3.62}$$

Combining the equation (3.30) and properties (2.13), (2.16) of Π_h, we obtain

$$B((\eta_u)_t,(\eta_p)_t,(\eta_q)_t,(\eta_r)_t;(\zeta_u)_t,(\zeta_r)_t,(\zeta_q)_t,-(\zeta_p)_t))$$
$$=\int_\Omega\left((\eta_u)_{tt}(\zeta_u)_t+(\eta_p)_t(\zeta_r)_t+(\eta_q)_t(\zeta_q)_t-(\eta_r)_t(\zeta_p)_t\right)dx. \tag{3.63}$$

Thus, we have

$$\int_\Omega\left((\zeta_u)_{tt}(\zeta_u)_t+(\zeta_q)_t^2\right)dx+\sum_j\left(\beta_1([(\zeta_q)_t]^2+[(\zeta_u)_t]^2)+\beta_2([(\zeta_p)_t]^2+[(\zeta_r)_t]^2)\right)_{j-\frac{1}{2}}$$
$$=-\int_\Omega\left((\eta_u)_{tt}(\zeta_u)_t+(\eta_p)_t(\zeta_r)_t+(\eta_q)_t(\zeta_q)_t-(\eta_r)_t(\zeta_p)_t\right)dx. \tag{3.64}$$

We sum up (3.53), (3.56), (3.58), (3.61) and (3.64), and get

$$\frac{1}{4}\frac{d}{dt}\int_\Omega\left(2\zeta_u^2+\zeta_p^2+\zeta_q^2+\zeta_r^2+2(\zeta_u)_t^2\right)dx+\frac{1}{2}\int_\Omega\left(2\zeta_q^2+\zeta_p^2+(\zeta_u)_t^2+(\zeta_r)_t^2+2(\zeta_q)_t^2\right)dx$$

$$+\sum_j \Big(\beta_1\Lambda([\zeta_u],[(\zeta_u)_t],[\zeta_q],[(\zeta_q)_t]) + \beta_2\Lambda([(\zeta_p)_t],[\zeta_p],[\zeta_r],[(\zeta_r)_t])\Big)_{j-\frac{1}{2}}$$
$$= P_1 + P_2 + Q_1 + Q_2 + R_1 + R_2 + T_1 + T_2. \tag{3.65}$$

Here $\Lambda(a,b,c,d)$ is a non-negative quadratic form as in (3.39), while $P_1, P_2, Q_1, Q_2, R_1, R_2, T_1, T_2$ are defined and bounded as below,

$$P_1 = -\int_\Omega (\frac{1}{2}\eta_p - \eta_r + \frac{1}{2}(\eta_p)_t - \frac{1}{2}(\eta_r)_t)\zeta_p dx \le Ch^{2k+2} + \frac{1}{4}\int_\Omega \zeta_p^2 dx, \quad P_2 = \int_\Omega (\eta_r)_t(\zeta_p)_t dx,$$
$$Q_1 = -\int_\Omega (\eta_q - \frac{1}{2}(\eta_u)_t + \frac{1}{2}(\eta_q)_t)\zeta_q dx \le Ch^{2k+2} + \frac{5}{4}\int_\Omega \zeta_q^2 dx,$$
$$Q_2 = -\int_\Omega ((\eta_q)_t - \frac{1}{2}(\eta_u)_t)(\zeta_q)_t dx \le Ch^{2k+2} + \int_\Omega (\zeta_q)_t^2 dx,$$
$$R_1 = -\int_\Omega ((\eta_p + \frac{1}{2}(\eta_r)_t)\zeta_r) dx \le Ch^{2k+2} + \frac{1}{4}\int_\Omega \zeta_r^2 dx,$$
$$R_2 = -\int_\Omega (\frac{1}{2}(\eta_r)_t + \frac{1}{2}\eta_p + (\eta_p)_t)(\zeta_r)_t dx \le Ch^{2k+2} + \frac{1}{2}\int_\Omega (\zeta_r)_t^2 dx,$$
$$T_1 = -\int_\Omega (\eta_u)_t \zeta_u dx \le Ch^{2k+2} + \frac{1}{2}\int_\Omega \zeta_u^2 dx,$$
$$T_2 = -\frac{1}{2}\int_\Omega (\eta_q + (\eta_q)_t + (\eta_u)_t + 2(\eta_u)_{tt})(\zeta_u)_t dx \le Ch^{2k+2} + \int_\Omega (\zeta_u)_t^2 dx.$$

With $\beta_1 \ge 0, \beta_2 \ge 0$ and the inequalities above, we get

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\int_\Omega \left(2\zeta_u^2 + \zeta_p^2 + \zeta_q^2 + \zeta_r^2 + 2(\zeta_u)_t^2\right) dx \\ &\le Ch^{2k+2} + \frac{1}{4}\int_\Omega (2\zeta_u^2 + \zeta_p^2 + \zeta_q^2 + \zeta_r^2 + 2(\zeta_u)_t^2) dx + P_2.\end{aligned} \tag{3.66}$$

For P_2, we have

$$\int_0^T P_2 dt = \int_\Omega (\eta_r)_t \zeta_p dx \,|_0^T - \int_0^T \int_\Omega (\eta_r)_{tt}\zeta_p dx dt \le Ch^{2k+2} + \frac{1}{8}\|\zeta_p\|_\Omega^2(T) + \frac{3}{8}\int_0^T \int_\Omega \zeta_p^2 dx dt.$$

Integrating in time for (3.66) over $[0,T]$, we obtain

$$\begin{aligned}&\frac{1}{4}\int_\Omega \left(2\zeta_u^2 + \frac{1}{2}\zeta_p^2 + \zeta_q^2 + \zeta_r^2 + 2(\zeta_u)_t^2\right) dx|_{t=T} \\ \le& Ch^{2k+2} + \frac{1}{4}\int_0^T \int_\Omega \left(2\zeta_u^2 + \frac{1}{2}\zeta_p^2 + \zeta_q^2 + \zeta_r^2 + 2(\zeta_u)_t^2\right) dx dt.\end{aligned} \tag{3.67}$$

Here we used the approximation property (2.14) of Π_h as well as the optimal error of the initialization in Lemma 3.7, and the constant C depends on k, the final time T, $\|u\|_{L^\infty((0,T);H^{k+4}(\Omega))}$, $\|u_t\|_{L^\infty((0,T);H^{k+4}(\Omega))}$, and $\|u_{tt}\|_{L^\infty((0,T);H^{k+2}(\Omega))}$.

Finally we apply the Gronwall's inequality and the triangle inequality, and reach the following theorem.

Theorem 3.8. *For the semi-discrete DG scheme* (3.24)-(3.27) *with the numerical fluxes* (3.28)*, where the parameters satisfy* $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$ *and* $\beta_i \geq 0$*,* $i = 1, 2$*, the following error estimates hold when the exact solution* u *of the equation* (1.2) *is sufficiently smooth,*

$$\|u - u_h\|_\Omega^2 + \|p - p_h\|_\Omega^2 + \|q - q_h\|_\Omega^2 + \|r - r_h\|_\Omega^2 + \|u_t - (u_h)_t\|_\Omega^2 \leq Ch^{2k+2}. \quad (3.68)$$

Here $r = u_x$*,* $q = r_x$*, and* $p = q_x$*. And the constant* C *depends on* k*, the final time* T*,* $\|u\|_{L^\infty((0,T);H^{k+4}(\Omega))}$*,* $\|u_t\|_{L^\infty((0,T);H^{k+4}(\Omega))}$*, and* $\|u_{tt}\|_{L^\infty((0,T);H^{k+2}(\Omega))}$*.*

3.3 Extension to general even-order equations

The DG methods with the special family of numerical fluxes in the previous Section 3.2, as well as the theoretical analysis for stability and optimal error estimates can be extended to the general even-order PDEs in (1.3). The key lies in a careful choice of numerical fluxes. In this subsection, we will particularly give the formulation of the methods as well as the theoretical results for the sixth order equation (1.3) with $n = 6$. Consider

$$u_t - u_x^{(6)} = 0, \quad (3.69)$$

with the periodic boundary condition. We first rewrite (3.69) into a first order system

$$u_t - u_x^5 = 0,\ u^5 = u_x^4,\ u^4 = u_x^3,\ u^3 = u_x^2,\ u^2 = u_x^1,\ u^1 = u_x, \quad (3.70)$$

then apply a DG method: look for $(u_h, u_h^5, u_h^4, u_h^3, u_h^2, u_h^1)$ with $u_h \in \mathcal{V}_h^k, u_h^i \in \mathcal{V}_h^k, i = 1, \cdots, 5$ such that for any $(v, v^5, v^4, v^3, v^2, v^1)$ with $v \in \mathcal{V}_h^k, v^i \in \mathcal{V}_h^k, i = 1, \cdots, 5$ and j

$$((u_h)_t, v)_{I_j} + (u_h^5, v_x)_{I_j} - (\widehat{u_h^5} v^-)_{j+\frac{1}{2}} + (\widehat{u_h^5} v^+)_{j-\frac{1}{2}} = 0, \quad (3.71)$$

$$(u_h^5, v^5)_{I_j} + (u_h^4, v_x^5)_{I_j} - (\widehat{u_h^4}(v^5)^-)_{j+\frac{1}{2}} + (\widehat{u_h^4}(v^5)^+)_{j-\frac{1}{2}} = 0, \quad (3.72)$$

$$(u_h^4, v^4)_{I_j} + (u_h^3, v_x^4)_{I_j} - (\widehat{u_h^3}(v^4)^-)_{j+\frac{1}{2}} + (\widehat{u_h^3}(v^4)^+)_{j-\frac{1}{2}} = 0, \quad (3.73)$$

$$(u_h^3, v^3)_{I_j} + (u_h^2, v_x^3)_{I_j} - (\widehat{u_h^2}(v^3)^-)_{j+\frac{1}{2}} + (\widehat{u_h^2}(v^3)^+)_{j-\frac{1}{2}} = 0, \quad (3.74)$$

$$(u_h^2, v^2)_{I_j} + (u_h^1, v_x^2)_{I_j} - (\widehat{u_h^1}(v^2)^-)_{j+\frac{1}{2}} + (\widehat{u_h^1}(v^2)^+)_{j-\frac{1}{2}} = 0, \quad (3.75)$$

$$(u_h^1, v^1)_{I_j} + (u_h, v_x^1)_{I_j} - (\widehat{u_h}(v^1)^-)_{j+\frac{1}{2}} + (\widehat{u_h}(v^1)^+)_{j-\frac{1}{2}} = 0. \quad (3.76)$$

Here, $\widehat{u_h}$ and $\widehat{u_h^i}, i = 1, \cdots, 5$ are numerical fluxes defined as

$$\widehat{u_h^5} = F_1(u_h^5, u_h, \beta_1),\ \widehat{u_h^4} = F_2(u_h^4, u_h^1, -\beta_2),\ \widehat{u_h^3} = F_1(u_h^3, u_h^2, \beta_1),$$
$$\widehat{u_h^2} = F_2(u_h^2, u_h^3, \beta_2),\ \widehat{u_h^1} = F_1(u_h^1, u_h^4, -\beta_1),\ \widehat{u_h} = F_2(u_h, u_h^5, \beta_2). \quad (3.77)$$

Under the similar assumptions for the parameters α, β_1, β_2 in the numerical fluxes as for the fourth order problem in Section 3.2, and following the similar analysis and definitions of initial conditions, one can carry out the L^2 stability and optimal error estimates of the DG methods above. The next two theorems summarize the results without the proofs.

Theorem 3.9. *With $\beta_1 \geq 0, \beta_2 \geq 0$, the semi-discrete DG scheme* (3.71)-(3.76) *with the numerical fluxes* (3.77) *for the sixth order equation* (1.3) *satisfies*

$$\|\boldsymbol{\mathcal{E}_h}(T)\|^2 \leq \|\boldsymbol{\mathcal{E}_h}(0)\|^2,$$

where $\|\boldsymbol{\mathcal{E}_h}(T)\|^2 = (8\|u_h\|_\Omega^2 + \|u_h^1\|_\Omega^2 + \|u_h^2\|_\Omega^2 + \|u_h^3\|_\Omega^2 + \|u_h^4\|_\Omega^2 + \|u_h^5\|_\Omega^2 + 8\|(u_h)_t\|_\Omega^2)\,|_{t=T}$.

Theorem 3.10. *For the semi-discrete DG scheme* (3.71)-(3.76) *with the numerical fluxes* (3.77), *where the parameters satisfy $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$ and $\beta_i \geq 0$, $i = 1, 2$, the following error estimates hold when the exact solution u of the equation* (1.3) *with $n = 6$ is sufficiently smooth,*

$$\begin{aligned}&\|u - u_h\|_\Omega^2 + \|u^1 - u_h^1\|_\Omega^2 + \|u^2 - u_h^2\|_\Omega^2 + \|u^3 - u_h^3\|_\Omega^2 + \|u^4 - u_h^4\|_\Omega^2\\ &+ \|u^5 - u_h^5\|_\Omega^2 + \|u_t - (u_h)_t\|_\Omega^2 \leq Ch^{2k+2}.\end{aligned} \tag{3.78}$$

Here $u^i = u_x^{(i-1)}, i = 1, 2, \cdots, 5$. And the constant C depends on k, the final time T, $\|u\|_{L^\infty((0,T);H^{k+6}(\Omega))}$, $\|u_t\|_{L^\infty((0,T);H^{k+6}(\Omega))}$, $\|u_{tt}\|_{L^\infty((0,T);H^{k+6}(\Omega))}$.

4 DG methods for the third order wave equation

In this section, we will propose a family of DG methods for the third order wave equation (1.4), with the optimally accurate LDG method in [30] as a special case. Following the analysis in [30], we will establish the L^2 stability and optimal error estimates of a sub-family of the proposed methods when the boundary conditions are periodic.

To formulate the DG methods for (1.4), we start with the first order form of the equation,

$$u_t + p_x = 0, \quad p - q_x = 0, \quad q - u_x = 0. \tag{4.1}$$

Based on this system, a DG method is to find u_h, p_h, $q_h \in \mathcal{V}_h^k$ such that for any v, w, $z \in \mathcal{V}_h^k$ and for any j,

$$((u_h)_t, v)_{I_j} - (p_h, v_x)_{I_j} + \left(F_p v^-\right)_{j+\frac{1}{2}} - \left(F_p v^+\right)_{j-\frac{1}{2}} = 0, \tag{4.2}$$

$$(p_h, w)_{I_j} + (q_h, w_x)_{I_j} - \left(F_q w^-\right)_{j+\frac{1}{2}} + \left(F_q w^+\right)_{j-\frac{1}{2}} = 0, \tag{4.3}$$

$$(q_h, z)_{I_j} + (u_h, z_x)_{I_j} - \left(F_u z^-\right)_{j+\frac{1}{2}} + \left(F_u z^+\right)_{j-\frac{1}{2}} = 0. \tag{4.4}$$

Here, F_p, F_q and F_u in (4.2)-(4.4) are single-valued numerical fluxes and they could take very general forms. To avoid overwhelmingly too many parameters, in this paper, we choose

$$F_p = F_1(p_h, u_h, \beta_1), \quad F_q = q_h^+, \quad F_u = F_2(u_h, p_h, \beta_2), \tag{4.5}$$

that involve three parameters α, β_1, β_2. The conditions on these parameters will be further specified along with our analysis for stability and error estimates.

By summing up (4.2)-(4.4) over j, we reach a compact form of the scheme: look for $u_h, p_h, q_h \in \mathcal{V}_h^k$ such that

$$B(u_h, p_h, q_h; v, w, z) = 0, \quad \forall v, w, z \in \mathcal{V}_h^k, \tag{4.6}$$

where

$$\begin{aligned} B(u_h, p_h, q_h; v, w, z) = &\int_\Omega (u_h)_t v dx - \sum_j \left(\int_{I_j} p_h v_x dx + (F_p[v])_{j-\frac{1}{2}} \right) \\ &+ \int_\Omega p_h w dx + \sum_j \left(\int_{I_j} q_h w_x dx + (F_q[w])_{j-\frac{1}{2}} \right) \\ &+ \int_\Omega q_h z dx + \sum_j \left(\int_{I_j} u_h z_x dx + (F_u[z])_{j-\frac{1}{2}} \right). \end{aligned} \tag{4.7}$$

4.1 L^2 stability

In this subsection, we present the L^2 stability analysis for the DG scheme (4.6)-(4.7) with the numerical fluxes (4.5) under some assumptions on the parameters α, β_1 and β_2. Similar to the analysis for the LDG method in [30], we first obtain four energy equations, and then prove the L^2 stability for the numerical solution u_h and the auxiliary variables p_h, q_h. Note that just as for the fourth and sixth order equations, more than one energy equation is needed in order for us to later establish optimal error estimates.

• **The first energy equation.** To obtain the energy equation related to $\|u_h\|_\Omega$, we take the test functions $v = u_h$, $w = q_h$ and $z = -p_h$ in (4.6). Then, the following equality is obtained

$$0 = B(u_h, p_h, q_h; u_h, q_h, -p_h) = \int_\Omega (u_h)_t u_h dx - \sum_j \left(\int_{I_j} p_h (u_h)_x dx + (F_p[u_h])_{j-\frac{1}{2}} \right)$$

$$
\begin{aligned}
&+\int_\Omega p_h q_h dx + \sum_j \left(\int_{I_j} q_h (q_h)_x dx + (F_q[q_h])_{j-\frac{1}{2}} \right) \\
&- \int_\Omega q_h p_h dx - \sum_j \left(\int_{I_j} u_h (p_h)_x dx + (F_u[p_h])_{j-\frac{1}{2}} \right) \\
=&\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx + \sum_j \left(\frac{1}{2}[q_h]^2_{j-\frac{1}{2}} - H_j(u_h, p_h) \right). \quad (4.8)
\end{aligned}
$$

Here, we use the definition of H_j in (3.8). Combing (4.8) with (3.9), we have

$$
B(u_h, p_h, q_h; u_h, q_h, -p_h) = \frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx + \sum_j \left(\frac{1}{2}[q_h]^2 - \beta_1[u_h]^2 - \beta_2[p_h]^2 \right)_{j-\frac{1}{2}} = 0. \tag{4.9}
$$

• **The second energy equation.** In the next step, we take the time derivative in (4.3)-(4.4), sum them up with (4.2) over j and have

$$
B(u_h, (p_h)_t, (q_h)_t; v, w, z) = 0, \quad \forall v, w, z \in \mathcal{V}_h^k. \tag{4.10}
$$

By taking the test functions $v = -(q_h)_t$, $w = p_h$ and $z = (u_h)_t$ in (4.10), we obtain

$$
\begin{aligned}
0 =&B(u_h, (p_h)_t, (q_h)_t; -(q_h)_t, p_h, (u_h)_t) \\
=& -\int_\Omega (u_h)_t (q_h)_t dx + \sum_j \left(\int_{I_j} p_h (q_h)_{tx} dx + (F_p[(q_h)_t])_{j-\frac{1}{2}} \right) \\
&+ \int_\Omega (p_h)_t p_h dx + \sum_j \left(\int_{I_j} (q_h)_t (p_h)_x dx + ((F_q)_t[p_h])_{j-\frac{1}{2}} \right) \\
&+ \int_\Omega (q_h)_t (u_h)_t dx + \sum_j \left(\int_{I_j} (u_h)_t (u_h)_{tx} dx + ((F_u)_t[(u_h)_t])_{j-\frac{1}{2}} \right). \quad (4.11)
\end{aligned}
$$

Using the definition of F_p, F_q and F_u in (4.5), we have the second energy equation as

$$
\begin{aligned}
0 =&B(u_h, (p_h)_t, (q_h)_t; -(q_h)_t, p_h, (u_h)_t) = \frac{1}{2}\frac{d}{dt}\int_\Omega p_h^2 dx \quad (4.12) \\
&+ \sum_j \left(-\alpha[(u_h)_t]^2 + (\alpha + \frac{1}{2})[p_h][(q_h)_t] + \beta_1[u_h][(q_h)_t] + \beta_2[(p_h)_t][(u_h)_t] \right)_{j-\frac{1}{2}}.
\end{aligned}
$$

• **The third energy equation.** Taking the time derivative in (4.2)-(4.4) and summing them up over j, we get

$$
B((u_h)_t, (p_h)_t, (q_h)_t; v, w, z) = 0, \quad \forall v, w, z \in \mathcal{V}_h^k. \tag{4.13}
$$

Similar to equation (4.8), we take $v = (u_h)_t$, $w = (q_h)_t$ and $z = -(p_h)_t$ in (4.13). Using the definition of F_p, F_q and F_u in (4.5) and H_j in (3.8), we obtain

$$\begin{aligned}
0 =& B((u_h)_t, (p_h)_t, (q_h)_t; (u_h)_t, (q_h)_t, -(p_h)_t) \\
=& \frac{1}{2}\frac{d}{dt}\int_\Omega (u_h)_t^2 dx + \sum_j \left(\frac{1}{2}[(q_h)_t^2]_{j-\frac{1}{2}} - H_j((u_h)_t, (p_h)_t) \right) \\
=& \frac{1}{2}\frac{d}{dt}\int_\Omega (u_h)_t^2 dx + \sum_j \left(\frac{1}{2}[(q_h)_t]^2 - \beta_1[(u_h)_t]^2 - \beta_2[(p_h)_t]^2 \right)_{j-\frac{1}{2}}.
\end{aligned} \tag{4.14}$$

• **The fourth energy equation.** In this step, we take the time derivative in (4.4), sum it up with (4.2)-(4.3) over j, and have

$$B(u_h, p_h, (q_h)_t; v, w, z) = 0, \quad \forall v, w, z \in \mathcal{V}_h^k. \tag{4.15}$$

By taking the test function $v = 0$, $w = \frac{1}{2}(u_h)_t$ and $z = \frac{1}{2}q_h$ in (4.15), we obtain

$$\begin{aligned}
0 =& B(u_h, p_h, (q_h)_t; 0, \frac{1}{2}(u_h)_t, \frac{1}{2}q_h) \\
=& \frac{1}{2}\int_\Omega p_h (u_h)_t dx + \frac{1}{2}\sum_j \left(\int_{I_j} q_h (u_h)_{tx} dx + (F_q[(u_h)_t])_{j-\frac{1}{2}} \right) \\
&+ \frac{1}{2}\int_\Omega (q_h)_t q_h dx + \frac{1}{2}\sum_j \left(\int_{I_j} (u_h)_t (q_h)_x dx + ((F_u)_t[q_h])_{j-\frac{1}{2}} \right).
\end{aligned} \tag{4.16}$$

Using the definitions of F_p, F_q and F_u in (4.5), we finally have

$$\begin{aligned}
0 = B(u_h, p_h, (q_h)_t; 0, \frac{1}{2}(u_h)_t, \frac{1}{2}q_h) =& \frac{1}{2}\int_\Omega (p_h (u_h)_t + (q_h)_t q_h) dx \\
&+ \frac{1}{2}\sum_j \left((\frac{1}{2} - \alpha)[q_h][(u_h)_t] + \beta_2[(p_h)_t][q_h] \right)_{j-\frac{1}{2}}.
\end{aligned} \tag{4.17}$$

Now we are ready to state the L^2 stability of the proposed DG methods.

Theorem 4.1. *Under the conditions*

$$\begin{aligned}
&\alpha^2 + \beta_1\beta_2 = \frac{1}{4}, \quad -2 \le \beta_1 \le 0, \quad -8 \le \beta_2 \le 0, \\
&4\beta_1 + 4\alpha \le \beta_2, \quad \alpha + 2\beta_2 \le -\frac{1}{2}, \quad 8\beta_1 + 7\alpha \le -\frac{1}{2} + \frac{3}{2}\beta_2,
\end{aligned} \tag{4.18}$$

the semi-discrete DG method (4.6)-(4.7) *with the numerical fluxes* (4.5) *satisfies*

$$\|u_h(T)\|_\Omega^2 + \|p_h(T)\|_\Omega^2 + \|(u_h)_t(T)\|_\Omega^2 + \frac{1}{2}\|q_h(T)\|_\Omega^2$$

$$\leq e^{\frac{T}{2}} \left(\|u_h(0)\|_\Omega^2 + \|p_h(0)\|_\Omega^2 + \|(u_h)_t(0)\|_\Omega^2 + \frac{1}{2}\|q_h(0)\|_\Omega^2 \right), \tag{4.19}$$

where T is the final time. Particularly, when $\alpha = -\frac{1}{2}$ and $\beta_1 = \beta_2 = 0$, the stability result (4.19) *can be replaced by*

$$\begin{aligned} &\|u_h(T)\|_\Omega^2 + \|p_h(T)\|_\Omega^2 + \|(u_h)_t(T)\|_\Omega^2 + \frac{1}{2}\|q_h(T)\|_\Omega^2 \\ &\leq \|u_h(0)\|_\Omega^2 + (\frac{T}{2}+1)\|p_h(0)\|_\Omega^2 + (\frac{T}{2}+1)\|(u_h)_t(0)\|_\Omega^2 + \frac{1}{2}\|q_h(0)\|_\Omega^2. \end{aligned} \tag{4.20}$$

Proof. Summing up the four energy equations (4.9), (4.12), (4.14) and (4.17), we have

$$\begin{aligned} 0 =& \frac{1}{2}\frac{d}{dt}\left(\|u_h\|_\Omega^2 + \|p_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2 + \frac{1}{2}\|q_h\|_\Omega^2 \right) + \frac{1}{2}(p_h, (u_h)_t) \\ &+ \sum_j \left(\frac{1}{2}[q_h]^2 + (-\alpha-\beta_1)[(u_h)_t]^2 + \frac{1}{2}[(q_h)_t]^2 - \beta_1[u_h]^2 - \beta_2[p_h]^2 - \beta_2[(p_h)_t]^2 \right)_{j-\frac{1}{2}} \\ &+ \sum_j \left((\alpha+\frac{1}{2})[p_h][(q_h)_t] + \beta_2[(p_h)_t][(u_h)_t] + \beta_1[u_h][(q_h)_t] \right)_{j-\frac{1}{2}} \\ &+ \frac{1}{2}\sum_j \left((\frac{1}{2}-\alpha)[q_h][(u_h)_t] + \beta_2[(p_h)_t][q_h] \right)_{j-\frac{1}{2}}. \end{aligned} \tag{4.21}$$

Now we introduce two symmetric matrices S_1 and S_2

$$S_1 = \begin{pmatrix} -\beta_1 & 0 & \frac{\beta_1}{2} \\ 0 & -\beta_2 & \frac{\alpha+\frac{1}{2}}{2} \\ \frac{\beta_1}{2} & \frac{\alpha+\frac{1}{2}}{2} & \frac{1}{2} \end{pmatrix}, \quad S_2 = \begin{pmatrix} \frac{1}{2} & \frac{\frac{1}{2}-\alpha}{4} & \frac{\beta_2}{4} \\ \frac{\frac{1}{2}-\alpha}{4} & -\alpha-\beta_1 & \frac{\beta_2}{2} \\ \frac{\beta_2}{4} & \frac{\beta_2}{2} & -\beta_2 \end{pmatrix} \tag{4.22}$$

and a set of vector-valued functions $\mathbf{U}_{1j}$ and $\mathbf{U}_{2j}$, $j = 1, \cdots, N$,

$$\mathbf{U}_{1j}^{\mathrm{T}} = ([u_h], [p_h], [(q_h)_t])_{j-\frac{1}{2}}, \qquad \mathbf{U}_{2j}^{\mathrm{T}} = ([q_h], [(u_h)_t], [(p_h)_t])_{j-\frac{1}{2}}. \tag{4.23}$$

Then, the equation (4.21) can be rewritten into

$$\begin{aligned} 0 =& \frac{1}{2}\frac{d}{dt}\left(\|u_h\|_\Omega^2 + \|p_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2 + \frac{1}{2}\|q_h\|_\Omega^2 \right) + \frac{1}{2}(p_h, (u_h)_t) \\ &+ \sum_j \left(\mathbf{U}_{1j}^{\mathrm{T}} S_1 \mathbf{U}_{1j} + \mathbf{U}_{2j}^{\mathrm{T}} S_2 \mathbf{U}_{2j} \right). \end{aligned} \tag{4.24}$$

In order to obtain conditions on α, β_1 and β_2 such that both S_1 and S_2 are positive semi-definite, we follow the sufficient and necessary condition "all the k-th principal minors are nonnegative, with $k = 1, 2, 3$", as well as the relation $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$ (to simplify the conditions), the details are given in Appendix B. This leads to (4.18), and under these conditions (4.24) becomes

$$\frac{1}{2}\frac{d}{dt}\left(\|u_h\|_\Omega^2 + \|p_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2 + \frac{1}{2}\|q_h\|_\Omega^2 \right) + \frac{1}{2}(p_h, (u_h)_t) \leq 0. \tag{4.25}$$

Now apply $|(p_h, (u_h)_t)| \le \frac{1}{2}(\|p_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2)$, and we can obtain

$$\frac{d}{dt}\left(\|u_h\|_\Omega^2 + \|p_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2 + \frac{1}{2}\|q_h\|_\Omega^2\right) \le \frac{1}{2}\left(\|p_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2\right) \tag{4.26}$$

$$\le \frac{1}{2}\left(\|u_h\|_\Omega^2 + \|p_h\|_\Omega^2 + \|(u_h)_t\|_\Omega^2 + \frac{1}{2}\|q_h\|_\Omega^2\right). \tag{4.27}$$

The stability result in (4.19) follows from the Gronwall's inequality.

Finally we consider a special case when $\alpha = -\frac{1}{2}$ and $\beta_1 = \beta_2 = 0$. Note that for this case, (4.12) and (4.14) imply

$$\|p_h(t)\|_\Omega \le \|p_h(0)\|_\Omega, \qquad \|(u_h)_t(t)\|_\Omega \le \|(u_h)_t(0)\|_\Omega, \ \ \forall t > 0. \tag{4.28}$$

Integrating (4.26) over $[0, T]$, we get

$$\|u_h(T)\|_\Omega^2 + \|p_h(T)\|_\Omega^2 + \|(u_h)_t(T)\|_\Omega^2 + \frac{1}{2}\|q_h(T)\|_\Omega^2 \tag{4.29}$$
$$\le \frac{1}{2}\int_0^T \left(\|p_h\|_\Omega^2(t) + \|(u_h)_t(t)\|_\Omega^2\right) dt + \|u_h(0)\|_\Omega^2 + \|p_h(0)\|_\Omega^2 + \|(u_h)_t(0)\|_\Omega^2 + \frac{1}{2}\|q_h(0)\|_\Omega^2.$$

The stability relation (4.20) follows from (4.29) and (4.28). □

Remark 4.2. When $\alpha = -\frac{1}{2}$ and $\beta_1 = \beta_2 = 0$, our proposed DG method will become the LDG method in [30] with one set of alternating numerical fluxes. And the stability result (4.20) was also established in [30].

We want to point out that the parameter conditions (4.18) do not include another set of alternating fluxes, that is, the numerical fluxes (4.5) with $\alpha = \frac{1}{2}, \beta_1 = \beta_2 = 0$, or equivalently, $F_p = p_h^+, F_q = q_h^+, F_u = u_h^-$. The corresponding DG method has quite different properties from that in [30] and also from those in Theorem 4.1, and its L^2 stability needs to be established separately. In the next Theorem, we state the energy stability result for this somewhat different DG method, and the proof is given in Appendix A.

Theorem 4.3. *Use the numerical fluxes* (4.5) *with* $\alpha = \frac{1}{2}, \beta_1 = \beta_2 = 0$*, the semi-discrete DG scheme* (4.6)-(4.7) *satisfies the energy stability*

$$\|u_h(T)\|_\Omega^2 + \frac{1}{2}\|p_h(T)\|_\Omega^2 + \|(u_h)_t(T)\|_\Omega^2 + \|q_h(T)\|_\Omega^2 + \|(q_h)_t(T)\|_\Omega^2 \tag{4.30}$$
$$\le \|u_h(0)\|_\Omega^2 + \frac{1}{2}\|p_h(0)\|_\Omega^2 + (\frac{T}{2}+1)\|(u_h)_t(0)\|_\Omega^2 + \|q_h(0)\|_\Omega^2 + (\frac{T}{2}+1)\|(q_h)_t(0)\|_\Omega^2.$$

4.2 L^2 error estimates

In this subsection, the optimal *a priori* L^2 error estimates will be proved for the DG method (4.6)-(4.7) for the third order equation when the exact solution is sufficiently smooth, under the conditions in (4.18) on the parameters α, β_1, β_2 in the numerical fluxes (4.5). Particularly, the relation $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$ holds. Since the numerical fluxes (4.5) are consistent, the exact solution u, $q = u_x$, and $p = q_x$ satisfy

$$B(u, p, q; v, w, z) = 0, \quad \forall v, w, z \in \mathcal{V}_h^k, \tag{4.31}$$

therefore we get the error equation

$$B(e_u, e_p, e_q; v, w, z) = 0, \ \forall v, w, z \in \mathcal{V}_h^k. \tag{4.32}$$

Here $e_\phi = \phi - \phi_h$, with $\phi = u, p, q$, are the error functions. In order to obtain the optimal error estimates, the following projection is used

$$\Pi_h \begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} \Pi_h^1(p, u, \beta_1) \\ \Pi_h^2(u, p, \beta_2) \end{pmatrix}, \tag{4.33}$$

and e_u, e_p will be decomposed into $e_\phi = \eta_\phi + \zeta_\phi, \phi = u, p$ based on (2.10)-(2.11), while $e_q = \eta_q + \zeta_q$ where $\eta_q = q - P_h^+ q$ and $\zeta_q = P_h^+ q - q_h$.

For the third order equation (1.4), we choose the initial condition $p_h(x, 0) = P_h^- p(x, 0)$ with $p(x, 0) = u_{xx}(x, 0)$. Based on $p_h(x, 0)$, we can further define the initial data $q_h(x, 0), u_h(x, 0) \in \mathcal{V}_h^k$, satisfying

$$(p_h, w)_{I_j} + (q_h, w_x)|_{I_j} - (\widehat{q}_h w^-)|_{j+\frac{1}{2}} + (\widehat{q}_h w^+)|_{j-\frac{1}{2}} = 0, \ \forall w \in \mathcal{V}_h^k,$$
$$\text{with } \widehat{q}_h = q_h^+(x, 0) \ \text{ and } \int_\Omega q_h(x, 0)dx = \int_\Omega q(x, 0)dx, \tag{4.34}$$

$$(q_h, w)_{I_j} + (u_h, w_x)|_{I_j} - (\widehat{u}_h w^-)|_{j+\frac{1}{2}} + (\widehat{u}_h w^+)|_{j-\frac{1}{2}} = 0, \ \forall w \in \mathcal{V}_h^k,$$
$$\text{with } \widehat{u}_h = u_h^+(x, 0) \ \text{ and } \int_\Omega u_h(x, 0)dx = \int_\Omega u(x, 0)dx. \tag{4.35}$$

Similar to the analysis for Lemma 5.1 in [23] and Lemma 2.4 in [30], the following lemma can be established.

Lemma 4.4. Assuming $u(x, 0)$ is sufficiently smooth, the initial conditions described above are well defined and satisfy the following estimates

$$||p(x, 0) - p_h(x, 0)||_\Omega \leq Ch^{k+1}, \ ||q(x, 0) - q_h(x, 0)||_\Omega \leq Ch^{k+1},$$
$$||u(x, 0) - u_h(x, 0)||_\Omega \leq Ch^{k+1}, \ ||u_t(x, 0) - (u_h)_t(x, 0)||_\Omega \leq Ch^{k+1}, \tag{4.36}$$

Here $(u_h)_t(x, 0)$ is determined by (4.2) with $F_p = p_h^-(x, 0)$ at $t = 0$. And C depends on $||u(x, 0)||_{H^{k+3}(\Omega)}$ and $q(x, 0) = u_x(x, 0)$.

To obtain the optimal error estimates, we follow the idea of the energy stability analysis and get four important error equations.

4.2.1 The first error equation

Since B is linear, the error equation (4.32) can be written as

$$B(\zeta_u, \zeta_p, \zeta_q; v, w, z) = -B(\eta_u, \eta_p, \eta_q; v, w, z), \quad \forall v, w, z \in \mathcal{V}_h^k. \tag{4.37}$$

We then take the test functions $v = \zeta_u$, $w = \zeta_q$ and $z = -\zeta_p$, all from $\mathcal{V}_h^k$, and get

$$B(\zeta_u, \zeta_p, \zeta_q; \zeta_u, \zeta_q, -\zeta_p) = \frac{1}{2}\frac{d}{dt}\int_\Omega \zeta_u^2 dx + \sum_j \left(\frac{1}{2}[\zeta_q]^2 - \beta_1[\zeta_u]^2 - \beta_2[\zeta_p]^2\right)_{j-\frac{1}{2}}, \tag{4.38}$$

$$\begin{aligned} B(\eta_u, \eta_p, \eta_q; \zeta_u, \zeta_q, -\zeta_p) &= \int_\Omega (\eta_u)_t \zeta_u dx - \sum_j \left(\int_{I_j} \eta_p (\zeta_u)_x dx + (F_1(\eta_p, \eta_u, \beta_1)[\zeta_u])_{j-\frac{1}{2}}\right) \\ &\quad + \int_\Omega \eta_p \zeta_q dx + \sum_j \left(\int_{I_j} \eta_q (\zeta_q)_x dx + ((\eta_q)^+[\zeta_q])_{j-\frac{1}{2}}\right) \\ &\quad - \int_\Omega \eta_q \zeta_p dx - \sum_j \left(\int_{I_j} \eta_u (\zeta_p)_x dx + (F_2(\eta_u, \eta_p, \beta_2)[\zeta_p])_{j-\frac{1}{2}}\right) \\ &= \int_\Omega \left((\eta_u)_t \zeta_u + \eta_p \zeta_q - \eta_q \zeta_p\right) dx. \end{aligned} \tag{4.39}$$

Here we have used (2.6) and the properties (2.13) and (2.16) in Lemma 2.1. Now combining (4.37)-(4.39), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \zeta_u^2 dx + \sum_j \left(\frac{1}{2}[\zeta_q]^2 - \beta_1[\zeta_u]^2 - \beta_2[\zeta_p]^2\right)_{j-\frac{1}{2}} \\ &= -\int_\Omega \left((\eta_u)_t \zeta_u + \eta_p \zeta_q - \eta_q \zeta_p\right) dx. \end{aligned} \tag{4.40}$$

4.2.2 The second error equation

Following the similar procedure to derive (4.10) in the stability analysis, we get an error equation in the following form,

$$B(\zeta_u, (\zeta_p)_t, (\zeta_q)_t; v, w, z) = -B(\eta_u, (\eta_p)_t, (\eta_q)_t; v, w, z), \quad \forall v, w, z \in \mathcal{V}_h^k. \tag{4.41}$$

Now we take the test functions $v = -(\zeta_q)_t$, $w = \zeta_p$ and $z = (\zeta_u)_t$, use the property (2.6) of P_h^+ and (2.13), (2.16) in Lemma 2.1, and obtain

$$\begin{aligned} &B(\zeta_u, (\zeta_p)_t, (\zeta_q)_t; -(\zeta_q)_t, \zeta_p, (\zeta_u)_t) = \frac{1}{2}\frac{d}{dt}\int_\Omega \zeta_p^2 dx \\ &+ \sum_j \left(-\alpha[(\zeta_u)_t]^2 + (\alpha + \frac{1}{2})[\zeta_p][(\zeta_q)_t] + \beta_1[\zeta_u][(\zeta_q)_t] + \beta_2[(\zeta_p)_t][(\zeta_u)_t]\right)_{j-\frac{1}{2}}, \end{aligned} \tag{4.42}$$

$$B(\eta_u,(\eta_p)_t,(\eta_q)_t;-(\zeta_q)_t,\zeta_p,(\zeta_u)_t)=\int_\Omega(-(\eta_u)_t(\zeta_q)_t+(\eta_p)_t\zeta_p+(\eta_q)_t(\zeta_u)_t)\,dx. \quad (4.43)$$

They, combined with (4.41), will lead to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\zeta_p^2dx+\sum_j\left(-\alpha[(\zeta_u)_t]^2+(\alpha+\frac{1}{2})[\zeta_p][(\zeta_q)_t]+\beta_1[\zeta_u][(\zeta_q)_t]+\beta_2[(\zeta_p)_t][(\zeta_u)_t]\right)_{j-\frac{1}{2}}$$
$$=-\int_\Omega(-(\eta_u)_t(\zeta_q)_t+(\eta_p)_t\zeta_p+(\eta_q)_t(\zeta_u)_t)\,dx. \quad (4.44)$$

4.2.3 The third error equation

Similar to how we derive (4.13), we can have the following error equation

$$B((\zeta_u)_t,(\zeta_p)_t,(\zeta_q)_t;v,w,z)=-B((\eta_u)_t,(\eta_p)_t,(\eta_q)_t;v,w,z),\quad\forall v,w,z\in\mathcal{V}_h^k. \quad (4.45)$$

With the test functions taken to be $v=(\zeta_u)_t$, $w=(\zeta_q)_t$ and $z=-(\zeta_p)_t$, the terms in (4.45) become

$$B((\zeta_u)_t,(\zeta_p)_t,(\zeta_q)_t;(\zeta_u)_t,(\zeta_q)_t,-(\zeta_p)_t)$$
$$=\frac{1}{2}\frac{d}{dt}\int_\Omega(\zeta_u)_t^2dx+\sum_j\left(\frac{1}{2}[(\zeta_q)_t]^2-\beta_1[(\zeta_u)_t]^2-\beta_2[(\zeta_p)_t]^2\right)_{j-\frac{1}{2}}, \quad (4.46)$$

$$B((\eta_u)_t,(\eta_p)_t,(\eta_q)_t;(\zeta_u)_t,(\zeta_q)_t,-(\zeta_p)_t)=\int_\Omega((\eta_u)_{tt}(\zeta_u)_t+(\eta_p)_t(\zeta_q)_t-(\eta_q)_t(\zeta_p)_t)\,dx. \quad (4.47)$$

Combining (4.45) with (4.46)-(4.47), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega(\zeta_u)_t^2dx+\sum_j\left(\frac{1}{2}[(\zeta_q)_t]^2-\beta_1[(\zeta_u)_t]^2-\beta_2[(\zeta_p)_t]^2\right)_{j-\frac{1}{2}}$$
$$=-\int_\Omega((\eta_u)_{tt}(\zeta_u)_t+(\eta_p)_t(\zeta_q)_t-(\eta_q)_t(\zeta_p)_t)\,dx. \quad (4.48)$$

4.2.4 The fourth error equation

Last, we use the error equation

$$B(\zeta_u,\zeta_p,(\zeta_q)_t;v,w,z)=-B(\eta_u,\eta_p,(\eta_q)_t;v,w,z),\quad\forall v,w,z\in\mathcal{V}_h^k. \quad (4.49)$$

Similar to the equation (4.16), we take the test functions $v=0$, $w=\frac{1}{2}(\zeta_u)_t$ and $z=\frac{1}{2}\zeta_q$ in (4.49), and obtain

$$B(\zeta_u,\zeta_p,(\zeta_q)_t;0,\frac{1}{2}(\zeta_u)_t,\frac{1}{2}\zeta_q)=\frac{1}{2}\int_\Omega(\zeta_p(\zeta_u)_t+(\zeta_q)_t\zeta_q)dx$$

$$+\frac{1}{2}\sum_j\left((\frac{1}{2}-\alpha)[\zeta_q][(\zeta_u)_t]+\beta_2[(\zeta_p)_t][\zeta_q]\right)_{j-\frac{1}{2}}, \quad (4.50)$$

and

$$B(\eta_u,\eta_p,(\eta_q)_t;0,\frac{1}{2}(\zeta_u)_t,\frac{1}{2}\zeta_q)=\frac{1}{2}\int_\Omega(\eta_p(\zeta_u)_t+(\eta_q)_t\zeta_q)\,dx, \quad (4.51)$$

hence we have

$$\frac{1}{2}\int_\Omega(\zeta_p(\zeta_u)_t+(\zeta_q)_t\zeta_q)dx+\frac{1}{2}\sum_j\left((\frac{1}{2}-\alpha)[\zeta_q][(\zeta_u)_t]+\beta_2[(\zeta_p)_t][\zeta_q]\right)_{j-\frac{1}{2}}$$
$$=-\frac{1}{2}\int_\Omega(\eta_p(\zeta_u)_t+(\eta_q)_t\zeta_q)\,dx. \quad (4.52)$$

We are now ready to establish the optimal error estimates for the proposed DG methods.

Theorem 4.5. *For the semi-discrete DG scheme* (4.6)-(4.7) *with the numerical fluxes* (4.5)*, where the parameters satisfy* (4.18)*, the following error estimates hold when the exact solution u of the equation* (1.4) *is sufficiently smooth,*

$$\|e_u\|_\Omega^2+\|e_p\|_\Omega^2+\|e_q\|_\Omega^2+\|(e_u)_t\|_\Omega^2\le Ch^{2k+2}. \quad (4.53)$$

Here $q=u_x, p=q_x$, and the constant C depends on k, the final time T, $\|u\|_{L^\infty((0,T);H^{k+3}(\Omega))}$, $\|u_t\|_{L^\infty((0,T);H^{k+3}(\Omega))}$ and $\|u_{tt}\|_{L^\infty((0,T);H^{k+3}(\Omega))}$.

Proof. Summing up the four error equations (4.40), (4.44), (4.48) and (4.52), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\zeta_u\|_\Omega^2+\|\zeta_p\|_\Omega^2+\|(\zeta_u)_t\|_\Omega^2+\frac{1}{2}\|\zeta_q\|_\Omega^2\right)+\sum_j\left(\mathbf{V}_{1j}^{\mathrm{T}}S_1\mathbf{V}_{1j}+\mathbf{V}_{2j}^{\mathrm{T}}S_2\mathbf{V}_{2j}\right)=\mathbf{F}+\mathbf{G},$$

where $\mathbf{V}_{1j}^{\mathrm{T}}=([\zeta_u],[\zeta_p],[(\zeta_q)_t])_{j-\frac{1}{2}}$, $\mathbf{V}_{2j}^{\mathrm{T}}=([\zeta_q],[(\zeta_u)_t],[(\zeta_p)_t])_{j-\frac{1}{2}}$ with $j=1,\cdots,N$ and

$$\mathbf{F}=-\int_\Omega\Big((\eta_u)_t\zeta_u+\eta_p\zeta_q-\zeta_p\eta_q+(\eta_p)_t\zeta_p+(\eta_q)_t(\zeta_u)_t+(\eta_u)_{tt}(\zeta_u)_t$$
$$+\frac{1}{2}\eta_p(\zeta_u)_t+\frac{1}{2}(\eta_q)_t\zeta_q\Big)dx-\frac{1}{2}\int_\Omega\zeta_p(\zeta_u)_tdx, \quad (4.54)$$

$$\mathbf{G}=-\int_\Omega((\eta_p)_t(\zeta_q)_t-(\zeta_p)_t(\eta_q)_t-(\eta_u)_t(\zeta_q)_t)\,dx. \quad (4.55)$$

For $\mathbf{F}$, we can bound it by

$$|\mathbf{F}|\le Ch^{2k+2}+\frac{1}{2}\left(\|\zeta_u\|_\Omega^2+\frac{1}{4}\|\zeta_q\|_\Omega^2+\frac{3}{4}\|\zeta_p\|_\Omega^2+\|(\zeta_u)_t\|_\Omega^2\right). \quad (4.56)$$

For $\mathbf{G}$, we integrate it in t over $[0,T]$, apply an integration by parts, and get

$$\begin{aligned}\int_0^T \mathbf{G} dt =& \int_\Omega (((\eta_u)_t - (\eta_p)_t)\zeta_q + (\eta_q)_t\zeta_p)\, dx \,|_0^T - \int_0^T\int_\Omega (((\eta_u)_{tt} - (\eta_p)_{tt})\zeta_q + (\eta_q)_{tt}\zeta_p)\, dxdt \\ \leq& Ch^{2k+2} + \frac{1}{4}\left(\|\zeta_p\|_\Omega^2 + \frac{1}{2}\|\zeta_q\|_\Omega^2\right)|_{t=T} + \frac{1}{8}\int_0^T \left(\|\zeta_p\|_\Omega^2 + \|\zeta_q\|_\Omega^2\right) dt. \qquad (4.57)\end{aligned}$$

Here, we have used the property of projections Π_h and P_h^+, as well as the optimal initial error estimates in Lemma 4.4.

Combining the two estimates above, we have

$$\begin{aligned}\int_0^T (\mathbf{F}+\mathbf{G})dt \leq& Ch^{2k+2} + \frac{1}{2}\int_0^T \left(\|\zeta_u\|_\Omega^2 + \frac{1}{2}\|\zeta_q\|_\Omega^2 + \|\zeta_p\|_\Omega^2 + \|(\zeta_u)_t\|_\Omega^2\right) dt \\ &+ \left(\frac{1}{4}\|\zeta_p\|_\Omega^2 + \frac{1}{8}\|\zeta_q\|_\Omega^2\right)|_{t=T}. \qquad (4.58)\end{aligned}$$

Recall that S_1 and S_2 are positive semi-definite under the conditions (4.18), and this further gives

$$\begin{aligned}&\frac{1}{2}\left(\|\zeta_u\|_\Omega^2 + \frac{1}{2}\|\zeta_q\|_\Omega^2 + \|\zeta_p\|_\Omega^2 + \|(\zeta_u)_t\|_\Omega^2\right)|_{t=T} \qquad (4.59)\\ \leq& Ch^{2k+2} + \frac{1}{2}\int_0^T \left(\|\zeta_u\|_\Omega^2 + \frac{1}{2}\|\zeta_q\|_\Omega^2 + \|\zeta_p\|_\Omega^2 + \|(\zeta_u)_t\|_\Omega^2\right) dt + \left(\frac{1}{4}\|\zeta_p\|_\Omega^2 + \frac{1}{8}\|\zeta_q\|_\Omega^2\right)|_{t=T},\end{aligned}$$

and therefore

$$\begin{aligned}&\frac{1}{8}\left(\|\zeta_u\|_\Omega^2 + \|\zeta_q\|_\Omega^2 + \|\zeta_p\|_\Omega^2 + \|(\zeta_u)_t\|_\Omega^2\right)|_{t=T} \\ \leq& Ch^{2k+2} + \frac{1}{2}\int_0^T \left(\|\zeta_u\|_\Omega^2 + \|\zeta_q\|_\Omega^2 + \|\zeta_p\|_\Omega^2 + \|(\zeta_u)_t\|_\Omega^2\right) dt. \qquad (4.60)\end{aligned}$$

Now we can apply the Gronwall's inequality, and reach

$$\|\zeta_u\|_\Omega^2 + \|\zeta_q\|_\Omega^2 + \|\zeta_p\|_\Omega^2 + \|(\zeta_u)_t\|_\Omega^2 \leq Ch^{2k+2}. \qquad (4.61)$$

Throughout the proof, the constant C depends on k, the final time T, $\|u\|_{L^\infty((0,T);H^{k+3}(\Omega))}$, $\|u_t\|_{L^\infty((0,T);H^{k+3}(\Omega))}$ and $\|u_{tt}\|_{L^\infty((0,T);H^{k+3}(\Omega))}$. Finally, we can get the error estimates (4.53) by combining (4.61) with the projection errors (2.8) and (2.14). □

Remark 4.6. We point out that for the semi-discrete DG method with the alternating flux $\alpha = \frac{1}{2}, \beta_1 = \beta_2 = 0$ (associated with Theorem 4.3), we just get its suboptimal error estimates because of the difficulty to find the well-defined initial conditions satisfying $||(q_t)(x,0) - (q_h)_t(x,0)||_\Omega^2 \leq Ch^{k+1}$. Here, we omit the proof of error estimates.

5 DG methods for the linear Schrödinger equation

In this section, we consider the linear Schrödinger equation (1.5), and will formulate and analyze DG methods to solve it. Given the solution is complex-valued, throughout the section, the L^2 inner product and its induced norm

$$(w,v)_K = \int_K wv^* dK, \quad \|v\|_K = \sqrt{(v,v)_K}, \tag{5.1}$$

are used for complex-valued square integrable functions in a domain K, and we will also work with the complex-valued discrete space ${}_c\mathcal{V}_h^k$, defined as

$${}_c\mathcal{V}_h^k = \{v = r + is : r|_{I_j} \in P^k(I_j), s|_{I_j} \in P^k(I_j), \forall I_j \in \mathcal{T}_h\}. \tag{5.2}$$

To obtain the DG methods for (1.5), we start with the first order form of the equation,

$$iu_t + p_x = 0, \quad p - u_x = 0. \tag{5.3}$$

Based on (5.3), our proposed DG method is to look for $u_h, p_h \in {}_c\mathcal{V}_h^k$ such that for any $v, w \in {}_c\mathcal{V}_h^k$ and for any j,

$$i\int_{I_j} (u_h)_t v dx - \int_{I_j} p_h v_x dx + (F_p v^-)_{j+\frac{1}{2}} - (F_p v^+)_{j-\frac{1}{2}} = 0, \tag{5.4}$$

$$\int_{I_j} p_h w dx + \int_{I_j} u_h w_x dx - (F_u w^-)_{j+\frac{1}{2}} + (F_u w^+)_{j-\frac{1}{2}} = 0. \tag{5.5}$$

Here F_p and F_u are numerical fluxes, taken as

$$F_p(p_h, u_h) = \{p_h\} + \alpha[p_h] + i\beta_1[u_h], \quad F_u(u_h, p_h) = \{u_h\} - \alpha[u_h] + i\beta_2[p_h], \tag{5.6}$$

and the parameters α, β_1, β_2 are $O(1)$ and real-valued, and they will be specified later for stability and optimal accuracy. By summing up (5.4)-(5.5) over j, we obtain a compact form of the scheme: look for $u_h, p_h \in {}_c\mathcal{V}_h^k$ such that

$$B(u_h, p_h; v, w) = 0, \quad \forall v, w \in {}_c\mathcal{V}_h^k, \tag{5.7}$$

where

$$\begin{aligned} B(u_h, p_h; v, w) =& i\int_\Omega (u_h)_t v dx - \sum_j \left(\int_{I_j} p_h v_x dx + (F_p[v])_{j-\frac{1}{2}} \right) \\ &+ \int_\Omega p_h w dx + \sum_j \left(\int_{I_j} u_h w_x dx + (F_u[w])_{j-\frac{1}{2}} \right). \end{aligned} \tag{5.8}$$

5.1 L^2 stability

In this subsection, the L^2 stability is established for the DG method (5.7)-(5.8) with the numerical fluxes (5.6) under some assumptions on the parameters. The analysis relies on three energy equations.

• **The first energy equation.** First, we take the test functions $v = u_h^*$, $w = p_h^*$ in (5.7), and obtain

$$\begin{aligned} 0 = B(u_h, p_h; u_h^*, p_h^*) =& i\int_\Omega (u_h)_t u_h^* dx - \sum_j \left(\int_{I_j} p_h (u_h^*)_x dx + (F_p[u_h^*])_{j-\frac{1}{2}} \right) \\ &+ \int_\Omega p_h p_h^* dx + \sum_j \left(\int_{I_j} u_h (p_h^*)_x dx + (F_u[p_h^*])_{j-\frac{1}{2}} \right). \end{aligned} \tag{5.9}$$

We then subtract the conjugate of (5.9) from itself, and get

$$i\frac{d}{dt}\int_\Omega |u_h|^2 dx + 2i\mathrm{Im}\left(\sum_j \left(\int_{I_j} \left(p_h^*(u_h)_x + u_h(p_h^*)_x\right) dx + (F_u[p_h^*] + (F_p)^*[u_h])_{j-\frac{1}{2}} \right)\right) = 0.$$

This, together with the definition of the numerical fluxes in (5.6), leads to

$$\frac{d}{dt}\int_\Omega |u_h|^2 dx + 2\left(\sum_j (-\beta_1[u_h][u_h^*] + \beta_2[p_h][p_h^*])_{j-\frac{1}{2}} \right) = 0. \tag{5.10}$$

• **The second energy equation.** We here want to derive the energy equation for p_h. By taking the time derivative of (5.5), summing it up with (5.4) over j, we get

$$\begin{aligned} 0 = B(u_h, (p_h)_t; v, w) =& i\int_\Omega (u_h)_t v dx - \sum_j \left(\int_{I_j} p_h v_x dx + (F_p[v])_{j-\frac{1}{2}} \right) \\ &+ \int_\Omega (p_h)_t w dx + \sum_j \left(\int_{I_j} (u_h)_t w_x dx + ((F_u)_t[w])_{j-\frac{1}{2}} \right). \end{aligned} \tag{5.11}$$

With the test functions being $v = -(u_h^*)_t$ and $w = p_h^*$, (5.11) becomes

$$\begin{aligned} 0 = B(u_h, (p_h)_t; -(u_h^*)_t, p_h^*) =& -i\int_\Omega (u_h)_t (u_h^*)_t dx + \sum_j \left(\int_{I_j} p_h (u_h^*)_{tx} dx + (F_p[(u_h^*)_t])_{j-\frac{1}{2}} \right) \\ &+ \int_\Omega (p_h)_t p_h^* dx + \sum_j \left(\int_{I_j} (u_h)_t (p_h^*)_x dx + ((F_u)_t[p_h^*])_{j-\frac{1}{2}} \right). \end{aligned} \tag{5.12}$$

Taking the conjugate of (5.12) and summing it up with (5.12), we have

$$0 = \frac{d}{dt}\int_\Omega |p_h|^2 dx + \sum_j \left(\int_{I_j} (p_h (u_h^*)_{tx} + (u_h^*)_t (p_h)_x) dx + (F_p[(u_h)_t^*] + (F_u)_t^*[p_h])_{j-\frac{1}{2}} \right)$$

$$+\sum_j \left(\int_{I_j} (p_h^*(u_h)_{tx} + (u_h)_t(p_h^*)_x)dx + ((F_p)^*[(u_h)_t] + (F_u)_t[p_h^*])_{j-\frac{1}{2}} \right). \qquad (5.13)$$

Combining (5.13) with the definition of F_p, F_u in (5.6), we have

$$\frac{d}{dt}\int_\Omega |p_h|^2 dx + 2\mathrm{Im}\left(\sum_j (-\beta_1[u_h][(u_h^*)_t] + \beta_2[(p_h^*)_t][p_h])_{j-\frac{1}{2}} \right) = 0. \qquad (5.14)$$

• **The third energy equation.** We start with taking the time derivative of (5.7), and then follow a similar procedure as to derive the first energy equation, except the test functions being taken as $v = (u_h^*)_t$, $w = (p_h^*)_t$. This leads to the third energy equation,

$$\frac{d}{dt}\int_\Omega |(u_h)_t|^2 dx + 2\sum_j (-\beta_1[(u_h)_t][(u_h^*)_t] + \beta_2[(p_h)_t][(p_h^*)_t])_{j-\frac{1}{2}} = 0. \qquad (5.15)$$

By summing up the three energy equations (5.10), (5.14) and (5.15), we now have

$$\frac{d}{dt}\int_\Omega \left(|u_h|^2 + |p_h|^2 + |(u_h)_t|^2\right) dx - 2\beta_1 \left(\sum_j \left(|[u_h]|^2 + |[(u_h)_t]|^2 + \mathrm{Im}([u_h][(u_h^*)_t])\right)_{j-\frac{1}{2}} \right)$$
$$+ 2\beta_2 \left(\sum_j \left(|[p_h]|^2 + |[(p_h)_t]|^2 + \mathrm{Im}([(p_h^*)_t][p_h])\right)_{j-\frac{1}{2}} \right) = 0, \qquad (5.16)$$

and this readily give us the L^2 stability result in the next Theorem.

Theorem 5.1. *With* $\beta_1 \le 0$ *and* $\beta_2 \ge 0$, *the semi-discrete DG scheme* (5.7)-(5.8) *with the numerical fluxes* (5.6) *satisfies*

$$\|u_h(T)\|_\Omega^2 + \|p_h(T)\|_\Omega^2 + \|(u_h)_t(T)\|_\Omega^2 \le \|u_h(0)\|_\Omega^2 + \|p_h(0)\|_\Omega^2 + \|(u_h)_t(0)\|_\Omega^2. \quad (5.17)$$

Remark 5.2. Compared with the Lemma 4.3 in [30], the proof for Theorem 5.1 requires an additional energy equation for $(u_h)_t$ with the presence of the parameters β_1, β_2. This energy relation is also important in error estimates to control both p_h and $(u_h)_t$.

Remark 5.3. The proof for Theorem 5.1 is also different from the L^2 stability in Section 3.1.1 for the heat equation. Here we can not directly get the L^2 stability for p_h unless we also have the energy relation for $(u_h)_t$.

5.2 L^2 error estimates

In this subsection, we will prove the optimal *a prior* L^2 error estimate for the DG method (5.7)-(5.8) for the linear Schrödinger equation (1.5) when the exact solution is sufficiently smooth, under the following assumption for the parameters in the numerical fluxes (5.6),

$$\alpha^2 - \beta_1\beta_2 = \frac{1}{4}, \quad \beta_1 \leq 0, \ \beta_2 \geq 0. \tag{5.18}$$

Since the numerical fluxes (5.6) are consistent, the exact solution u and $p = u_x$ satisfy

$$B(u, p; v, w) = 0, \quad \forall v, w \in {}_c\mathcal{V}_h^k, \tag{5.19}$$

hence we get the error equation

$$B(e_u, e_p; v, w) = 0, \quad \forall v, w \in {}_c\mathcal{V}_h^k. \tag{5.20}$$

Here $e_\phi = \phi - \phi_h$, with $\phi = u, p$ are error functions. To ensure the error estimates to be optimal, we use a special projection,

$${}_c\Pi_h \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} P_h^+ \left((\frac{1}{2} - \alpha)u + i\beta_2 p\right) + P_h^- \left((\frac{1}{2} + \alpha)u - i\beta_2 p\right) \\ P_h^+ \left((\frac{1}{2} + \alpha)p + i\beta_1 u\right) + P_h^- \left((\frac{1}{2} - \alpha)p - i\beta_1 u\right) \end{pmatrix} \tag{5.21}$$

that maps from ${}_cH^1(\Omega) \times {}_cH^1(\Omega)$ onto ${}_c\mathcal{V}_h^k \times {}_c\mathcal{V}_h^k$, ${}_cH^1(\Omega)$ denotes the function space with the real and the imaginary parts in $H^1(\Omega)$. Using ${}_c\Pi_h$, the error functions can be decomposed into $e_\phi = \eta_\phi + \zeta_\phi, \phi = u, p$ based on (2.10)-(2.11), with Π_h replaced by ${}_c\Pi_h$.

The operator ${}_c\Pi_h$ is motivated by Π_h in (2.9) (also see [9]), and it is tailored for the numerical flux (5.6). Following a similar proof for Lemma 2.4 in [9], we can show the following Lemma.

Lemma 5.4. For any given α, β_1, β_2, the operator ${}_c\Pi_h$ has the following properties:

$$(i) \quad \int_{I_j} \eta_u \phi_x^* dx = 0, \quad \int_{I_j} \eta_p \psi_x^* dx = 0, \quad \forall \phi, \psi \in {}_c\mathcal{V}_h^k, \quad \forall j, \tag{5.22}$$

$$(ii) \quad \left\| \begin{pmatrix} u \\ p \end{pmatrix} - {}_c\Pi_h \begin{pmatrix} u \\ p \end{pmatrix} \right\| \leq C_* \left(1 + |\alpha| + \max(|\beta_1|, |\beta_2|)\right) h^{k+1} \left(\|u\|_{H^{k+1}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \right). \tag{5.23}$$

If we further assume $\alpha^2 - \beta_1\beta_2 = \frac{1}{4}$, we have

$$(iii) \quad {}_c\Pi_h \text{ defines a projection, that is } ({}_c\Pi_h)^2 = {}_c\Pi_h, \tag{5.24}$$

$$(iv) \quad F_p(\eta_p, \eta_u)_{j-\frac{1}{2}} = 0, \quad F_u(\eta_u, \eta_p)_{j-\frac{1}{2}} = 0, \quad \forall j. \tag{5.25}$$

Here F_p and F_u are defined in (5.6), and $(u, p) \in {}_cH^{k+1}(\Omega) \times {}_cH^{k+1}(\Omega)$.

We choose the initial condition $p_h(x,0) = P_h^- p(x,0)$ with $p(x,0) = u_x(x,0)$. Using $p_h(x,0)$, we can further define the initial data $u_h(x,0) \in \mathcal{V}_h^k$ which satisfies

$$(p_h, w)_{I_j} + (u_h, w_x)|_{I_j} - (\widehat{u}_h w^-)|_{j+\frac{1}{2}} + (\widehat{u}_h w^+)|_{j-\frac{1}{2}} = 0, \ \forall w \in \mathcal{V}_h^k,$$
$$\text{with } \widehat{u}_h = u_h^+(x,0) \ \text{ and } \ \int_\Omega u_h(x,0)dx = \int_\Omega u(x,0)dx. \quad (5.26)$$

Then, following the analysis for Lemma 5.1 in [23] and Lemma 2.4 in [30], we have the following estimates for the initial data.

Lemma 5.5. Assuming $u(x,0)$ is sufficiently smooth, the initial conditions described above are well defined and satisfy the following

$$\begin{aligned}&||p(x,0) - p_h(x,0)||_\Omega \le Ch^{k+1}, \ ||u(x,0) - u_h(x,0)||_\Omega \le Ch^{k+1},\\ &||u_t(x,0) - (u_h)_t(x,0)||_\Omega \le Ch^{k+1}.\end{aligned} \quad (5.27)$$

Here $(u_h)_t(x,0)$ is determined by (5.4) with $F_p = p_h^-(x,0)$ at $t = 0$, and C depends on $||u(x,0)||_{H^{k+2}(\Omega)}$.

To obtain the optimal error estimates, we follow the line of stability analysis and get three error equations.

5.2.1 The first error equation

Taking the test functions $v = \zeta_u^*$ and $w = \zeta_p^*$ in the error equation (5.20), we have

$$B(\zeta_u, \zeta_p; \zeta_u^*, \zeta_p^*) + B(\eta_u, \eta_p; \zeta_u^*, \zeta_p^*) = 0. \quad (5.28)$$

Now we follow a similar procedure to get the first energy equation (5.9), and use the definition of B in (5.8), and get

$$\begin{aligned} B(\zeta_u, \zeta_p; \zeta_u^*, \zeta_p^*) =& i\int_\Omega (\zeta_u)_t \zeta_u^* dx - \sum_j \left(\int_{I_j} \zeta_p (\zeta_u^*)_x dx + (F_p[\zeta_u^*])_{j-\frac{1}{2}} \right) \\ &+ \int_\Omega \zeta_p \zeta_p^* dx + \sum_j \left(\int_{I_j} \zeta_u (\zeta_p^*)_x dx + (F_u[\zeta_p^*])_{j-\frac{1}{2}} \right), \quad (5.29) \\ B(\eta_u, \eta_p; \zeta_u^*, \zeta_p^*) =& i\int_\Omega (\eta_u)_t \zeta_u^* dx + \int_\Omega \eta_p \zeta_p^* dx. \quad (5.30) \end{aligned}$$

Here, we have used the properties (5.22) and (5.25) of the projection ${}_c\Pi_h$ for (5.30). Now we subtract (5.28) by its conjugate, and use (5.29) and (5.30), this will lead to

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\zeta_u|^2 dx + 2\left(\sum_j \left(-\beta_1[\zeta_u][\zeta_u^*] + \beta_2[\zeta_p][\zeta_p^*] \right)_{j-\frac{1}{2}} \right) \\ &= -2\text{Re}\left(\int_\Omega (\eta_u)_t \zeta_u^* dx \right) - 2\text{Im}\left(\int_\Omega \eta_p \zeta_p^* dx \right). \quad (5.31) \end{aligned}$$

5.2.2 The second error equation

Following the second energy equation (5.11) and replacing u_h, p_h by e_u, e_p, we have

$$B(\zeta_u, (\zeta_p)_t; v, w) + B(\eta_u, (\eta_p)_t; v, w) = 0, \quad \forall v, w \in {}_c\mathcal{V}_h^k. \tag{5.32}$$

Taking the test functions $v = -(\zeta_u^*)_t$ and $w = \zeta_p^*$ in (5.32), we get

$$B(\zeta_u, (\zeta_p)_t; -(\zeta_u^*)_t, \zeta_p^*) = -B(\eta_u, (\eta_p)_t; -(\zeta_u^*)_t, \zeta_p^*), \tag{5.33}$$

with

$$\begin{aligned} B(\zeta_u, (\zeta_p)_t; -(\zeta_u^*)_t, \zeta_p^*) = &- i\int_\Omega (\zeta_u)_t(\zeta_u^*)_t dx + \sum_j \left(\int_{I_j} \zeta_p(\zeta_u^*)_{tx} dx + (F_p[(\zeta_u^*)_t])_{j-\frac{1}{2}}\right) \\ &+ \int_\Omega (\zeta_p)_t\zeta_p^* dx + \sum_j \left(\int_{I_j}(\zeta_u)_t(\zeta_p^*)_x dx + ((F_u)_t[\zeta_p^*])_{j-\frac{1}{2}}\right), \end{aligned} \tag{5.34}$$

$$B(\eta_u, (\eta_p)_t; -(\zeta_u^*)_t, \zeta_p^*) = - i\int_\Omega (\eta_u)_t(\zeta_u^*)_t dx + \int_\Omega (\eta_p)_t\zeta_p^* dx. \tag{5.35}$$

We now add (5.33) and its conjugate and get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\zeta_p|^2 dx + 2\text{Im}\left(\sum_j \left(-\beta_1[\zeta_u][(\zeta_u^*)_t] + \beta_2[(\zeta_p^*)_t][\zeta_p]\right)_{j-\frac{1}{2}}\right) \\ &\quad = -2\text{Im}\int_\Omega (\eta_u)_t(\zeta_u^*)_t dx - 2\text{Re}\int_\Omega (\eta_p)_t\zeta_p^* dx. \end{aligned} \tag{5.36}$$

Here, the definition of F_p, F_u in (5.6) and the properties (5.22), (5.25) of the projection ${}_c\Pi_h$ are used.

5.2.3 The third error equation

In the last step, we follow the third energy equation (5.15). Similar to the equation (5.28), the test functions are taken to be $v = (\zeta_u^*)_t, w = (\zeta_p^*)_t$. Then, we have

$$B((\zeta_u)_t, (\zeta_p)_t; (\zeta_u^*)_t, (\zeta_p^*)_t) = -B((\eta_u)_t, (\eta_p)_t; (\zeta_u^*)_t, (\zeta_p^*)_t). \tag{5.37}$$

We subtract (5.37) by its conjugate, and get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |(\zeta_u)_t|^2 dx + 2\left(\sum_j \left(-\beta_1[(\zeta_u)_t][(\zeta_u^*)_t] + \beta_2[(\zeta_p)_t][(\zeta_p^*)_t]\right)_{j-\frac{1}{2}}\right) \\ &= -2\text{Re}\left(\int_\Omega (\eta_u)_{tt}(\zeta_u^*)_t dx\right) - 2\text{Im}\left(\int_\Omega (\eta_p)_t(\zeta_p^*)_t dx\right). \end{aligned} \tag{5.38}$$

Once we have the three error equations, (5.31), (5.36) and (5.38), we sum them up and get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(|(\zeta_u)_t|^2 + |\zeta_p|^2 + |\zeta_u|^2\right) dx \\
&\quad - 2\beta_1 \sum_j \left(|[\zeta_u]|^2 + \mathrm{Im}([\zeta_u][(\zeta_u^*)_t]) + |[(\zeta_u)_t]|^2\right)_{j-\frac{1}{2}} \\
&\quad + 2\beta_2 \sum_j \left(|[\zeta_p]|^2 + \mathrm{Im}([\zeta_p][(\zeta_p^*)_t]) + |[(\zeta_p)_t]|^2\right)_{j-\frac{1}{2}} \\
&=2\mathrm{Re}(\Theta) + 2\mathrm{Im}(\Gamma) + 2\mathrm{Im}(\Lambda).
\end{aligned} \tag{5.39}$$

Here, Θ, Γ, Λ denote

$$\Theta = -\int_\Omega \left((\eta_u)_t\zeta_u^* + (\eta_p)_t\zeta_p^* + (\eta_u)_{tt}(\zeta_u^*)_t\right) dx, \tag{5.40}$$

$$\Gamma = -\int_\Omega \left(\eta_p\zeta_p^* + (\eta_u)_t(\zeta_u^*)_t\right) dx, \quad \Lambda = -\int_\Omega (\eta_p)_t(\zeta_p^*)_t dx. \tag{5.41}$$

Related to Λ, we have

$$\int_0^T \Lambda dt = -\int_\Omega ((\eta_p)_t\zeta_p^*)dx|_0^T + \int_0^T \int_\Omega (\eta_p)_{tt}\zeta_p^* dt, \tag{5.42}$$

hence

$$\int_0^T \mathrm{Im}(\Lambda)dt \le Ch^{2k+2} + \frac{1}{4}\int_\Omega |\zeta_p|^2 dx + \frac{1}{8}\int_0^T \int_\Omega |\zeta_p|^2 dxdt. \tag{5.43}$$

As for Θ and Γ, we have

$$\begin{aligned}
\mathrm{Re}(\Theta) \le& Ch^{2k+2} + \frac{1}{8}\int_\Omega (2|\zeta_u|^2 + |(\zeta_u)_t|^2 + \frac{1}{2}|\zeta_p|^2)dx, \\
\mathrm{Im}(\Gamma) \le& Ch^{2k+2} + \frac{1}{8}\int_\Omega (|(\zeta_u)_t|^2 + \frac{1}{2}|\zeta_p|^2),
\end{aligned} \tag{5.44}$$

thus

$$\int_0^T (\mathrm{Re}(\Theta) + \mathrm{Im}(\Gamma))dt \le Ch^{2k+2} + \frac{1}{4}\int_0^T \int_\Omega \left(|\zeta_u|^2 + |(\zeta_u)_t|^2 + \frac{1}{2}|\zeta_p|^2\right) dxdt. \tag{5.45}$$

Here, we have used the Young inequality and the optimal error estimates in Lemma 5.5 from the initialization. Now we combine (5.39)-(5.45) with $\beta_1 \le 0$ and $\beta_2 \ge 0$, and get

$$\int_\Omega \left(|(\zeta_u)_t|^2 + \frac{1}{2}|\zeta_p|^2 + |\zeta_u|^2\right)|_{t=T} dx \le Ch^{2k+2} \tag{5.46}$$

$$+\frac{1}{2}\int_0^T\int_\Omega\left(|\zeta_u|^2+|(\zeta_u)_t|^2+\frac{1}{2}|\zeta_p|^2\right)dxdt.$$

In this section, C depends on k, T, $\|u\|_{L^\infty((0,T);H^{k+2}(\Omega))}$, $\|u_t\|_{L^\infty((0,T);H^{k+2}(\Omega))}$ and $\|u_{tt}\|_{L^\infty((0,T);H^{k+2}(\Omega))}$. Finally we apply the Gronwall's inequality to (5.46), and reach the following theorem.

Theorem 5.6. *For the semi-discrete DG scheme* (5.7)-(5.8) *with the numerical fluxes* (5.6) *under the conditions in* (5.18)*, the following error estimate holds when the exact solution u of the equation* (1.5) *is sufficiently smooth,*

$$\|e_u\|_\Omega^2+\|(e_u)_t\|_\Omega^2+\|e_p\|_\Omega^2\le Ch^{2k+2}. \tag{5.47}$$

Here $p=u_x$, and C depends on $\|u\|_{L^\infty((0,T);H^{k+2}(\Omega))}$, $\|u_t\|_{L^\infty((0,T);H^{k+2}(\Omega))}$ and $\|u_{tt}\|_{L^\infty((0,T);H^{k+2}(\Omega))}$.

6 Numerical examples

In this section, we present numerical examples to demonstrate the performance of the proposed methods, and to verify our theoretical results in previous sections. In our numerical experiments, the implicit $(k+1)$-th order SDC method in [22, 28] is utilized as the time discretization for the DG methods when the P^k polynomial spaces are used. The implicit SDC temporal discretization allows the time step to be $\Delta t=O(h)$ and can be easily implemented to have arbitrary order of accuracy. In all numerical tests, uniform meshes with N cells are used.

6.1 The heat equation

In this subsection, we consider the one dimensional heat equation (1.1), with the initial data

$$u(x,0)=\sin(x),\quad \text{on} \quad (0,2\pi), \tag{6.1}$$

and the periodic boundary condition. The exact solution is $u(x,t)=e^{-t}\sin(x)$. We test the problem with different sets of α,β_1,β_2, and compute up to $T=5$ based on P^k approximations with $k=1,2,3$. The time step is taken as $\Delta t=h$. Table 1 and Table 2 show L^2 errors and orders of accuracy of the numerical solution u_h and the auxiliary variable p_h. We observe that both u_h and p_h from the proposed DG methods for the equation (1.1) are $(k+1)$-th order accurate with $k=1,2,3$ when the numerical fluxes (3.4) satisfy the conditions, $\alpha^2+\beta_1\beta_2=\frac{1}{4},\beta_1\ge0,\beta_2\ge0$, and this verifies our theoretical results. On the other hand, when the numerical fluxes are central (with $\alpha=\beta_1=\beta_2=0$), and it does not satisfies $\alpha^2+\beta_1\beta_2=\frac{1}{4}$, the numerical solutions are suboptimal when k is odd.

Table 1: L^2 errors and orders of accuracy of u_h for the heat equation (1.1) at time $T = 5$.

k	N	α=-0.5 β_1=β_2=0		α=0.5 β_1=β_2=0		α=-0.499 $\beta_1=\beta_2=\sqrt{0.25-0.499^2}$		α=-0.5 β_1=0, β_2=0.5		$\alpha=-\sqrt{0.25-0.06}$ β_1=0.2, β_2=0.3		α=0 β_1=β_2=0	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^1	10	2.20E-03	–	2.20E-03	–	2.22E-03	–	2.55E-03	–	2.34E-03	–	1.71E-03	–
	20	6.31E-04	1.80	6.31E-04	1.80	6.32E-04	1.81	6.73E-04	1.92	6.47E-04	1.86	6.29E-04	1.45
	40	1.72E-04	1.87	1.72E-04	1.87	1.72E-04	1.88	1.78E-04	1.92	1.74E-04	1.89	2.58E-04	1.29
	80	4.56E-05	1.92	4.56E-05	1.92	4.56E-05	1.92	4.66E-05	1.94	4.59E-05	1.93	1.18E-04	1.13
	160	1.18E-05	1.95	1.18E-05	1.95	1.18E-05	1.95	1.20E-05	1.96	1.19E-05	1.95	5.73E-05	1.04
	320	3.01E-06	1.97	3.01E-06	1.97	3.01E-06	1.97	3.04E-06	1.98	3.02E-06	1.98	2.84E-05	1.01
P^2	10	6.58E-04	–	6.58E-04	–	6.60E-04	–	6.99E-04	–	6.76E-04	–	6.57E-04	–
	20	8.38E-05	2.97	8.38E-05	2.97	8.39E-05	2.98	8.64E-05	3.02	8.47E-05	3.00	8.28E-05	2.99
	40	1.05E-05	3.00	1.05E-05	3.00	1.05E-05	3.00	1.06E-05	3.02	1.05E-05	3.01	1.03E-05	3.00
	80	1.31E-06	3.00	1.31E-06	3.00	1.32E-06	3.00	1.32E-06	3.01	1.30E-06	3.01	1.29E-06	3.00
	160	1.63E-07	3.00	1.63E-07	3.00	1.63E-07	3.00	1.64E-07	3.01	1.62E-07	3.01	1.61E-07	3.00
	320	2.04E-08	3.00	2.04E-08	3.00	2.04E-08	3.00	2.04E-08	3.00	2.02E-08	3.00	2.10E-08	3.00
P^3	10	1.83E-04	–	1.83E-04	–	1.84E-04	–	1.93E-04	–	1.88E-04	–	1.80E-04	–
	20	1.16E-05	3.98	1.16E-05	3.98	1.16E-05	3.98	1.20E-05	4.01	1.18E-05	4.00	1.44E-05	3.65
	40	7.26E-07	4.00	7.26E-07	4.00	7.26E-07	4.00	7.42E-07	4.01	7.31E-07	4.01	1.42E-06	3.34
	80	4.53E-08	4.00	4.53E-08	4.00	4.53E-08	4.00	4.61E-08	4.01	4.55E-08	4.01	1.64E-07	3.12
	160	2.83E-09	4.00	2.83E-09	4.00	2.83E-09	4.00	2.87E-09	4.00	2.84E-09	4.00	2.00E-08	3.03
	320	1.77E-10	4.00	1.77E-10	4.00	1.77E-10	4.00	1.79E-10	4.00	1.77E-10	4.00	2.49E-09	3.01

Table 2: L^2 errors and orders of accuracy of p_h with $p = u_x$ for the heat equation (1.1) at time $T = 5$.

k	N	α=-0.5 β_1=β_2=0		α=0.5 β_1=β_2=0		α=-0.499 $\beta_1=\beta_2=\sqrt{0.25-0.499^2}$		α=-0.5 β_1=0, β_2=0.5		$\alpha=-\sqrt{0.25-0.06}$ β_1=0.2, β_2=0.3		α=0 β_1=β_2=0	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^1	10	2.03E-03	–	2.03E-03	–	2.05E-03	–	2.31E-03	–	2.16E-03	–	1.74E-03	–
	20	6.30E-04	1.69	6.30E-04	1.69	6.28E-04	1.71	6.40E-04	1.85	6.22E-04	1.80	6.34E-04	1.46
	40	1.72E-04	1.87	1.72E-04	1.87	1.72E-04	1.87	1.73E-04	1.88	1.71E-04	1.86	2.59E-04	1.29
	80	4.56E-05	1.92	4.56E-05	1.92	4.56E-05	1.92	4.57E-05	1.92	4.54E-05	1.91	1.18E-04	1.13
	160	1.18E-05	1.95	1.18E-05	1.95	1.18E-05	1.95	1.18E-05	1.95	1.18E-05	1.95	5.73E-05	1.04
	320	3.01E-06	1.97	3.01E-06	1.97	3.01E-06	1.97	3.01E-06	1.97	3.00E-06	1.97	2.84E-05	1.01
P^2	10	6.68E-04	–	6.68E-04	–	6.63E-04	–	6.62E-04	–	6.48E-04	–	6.53E-04	–
	20	8.38E-05	2.97	8.38E-05	2.97	8.37E-05	2.98	8.40E-05	2.98	8.33E-05	3.00	8.28E-05	2.98
	40	1.05E-05	3.00	1.05E-05	3.00	1.05E-05	3.00	1.05E-05	3.00	1.05E-05	2.99	1.03E-05	3.00
	80	1.31E-06	3.00	1.31E-06	3.00	1.31E-06	3.00	1.31E-06	3.00	1.31E-06	3.00	1.29E-06	3.00
	160	1.63E-07	3.00	1.63E-07	3.00	1.63E-07	3.00	1.63E-07	3.01	1.64E-07	3.00	1.61E-07	3.00
	320	2.04E-08	3.00	2.04E-08	3.00	2.04E-08	3.00	2.04E-08	3.00	2.05E-08	3.00	2.01E-08	3.00
P^3	10	1.84E-04	–	1.84E-04	–	1.81E-04	–	1.84E-04	–	1.81E-04	–	1.81E-04	–
	20	1.16E-05	3.99	1.16E-05	3.99	1.16E-05	3.96	1.16E-05	3.99	1.15E-05	3.97	1.44E-05	3.65
	40	7.26E-07	4.00	7.26E-07	4.00	7.26E-07	4.00	7.26E-07	4.00	7.23E-07	3.99	1.42E-06	3.34
	80	4.53E-08	4.00	4.53E-08	4.00	4.53E-08	4.00	4.53E-08	4.00	4.52E-08	4.00	1.64E-07	3.12
	160	2.83E-09	4.00	2.83E-09	4.00	2.83E-09	4.00	2.83E-09	4.00	2.82E-09	4.00	2.00E-08	3.03
	320	1.77E-10	4.00	1.77E-10	4.00	1.77E-10	4.00	1.77E-10	4.00	1.76E-10	4.00	2.49E-09	3.01

6.2 The fourth order equation

Here, we consider the fourth order equation (1.2) with the initial condition

$$u(x,0) = \sin(x), \quad \text{on} \quad (0, 2\pi). \tag{6.2}$$

Periodic boundary condition is used. The exact solution is $u(x,t) = e^{-t}\sin(x)$. Several sets of the parameters α, β_1, β_2 are used in the numerical fluxes (3.28) to test the proposed DG methods, and the problem is computed up to $T = 5$ based on P^k approximation with $k = 1, 2, 3$. For this test, we use the time step $\Delta t = h$. The numerical results are shown in Table 3 for the L^2 errors and orders of accuracy of the numerical solution u_h, and they confirm the $(k+1)$-th order of accuracy for u with $k = 1, 2, 3$, when the parameter set satisfies $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}, \beta_1 \geq 0, \beta_2 \geq 0$. The DG method with the central flux, which does not satisfy the condition above, yields the suboptimal rate when k is odd.

In Table 4, we present the L^2 errors and orders of accuracy of the auxiliary variables r_h, q_h, p_h with $r = u_x, q = r_x, p = q_x$ from the DG methods which use the numerical fluxes (3.28) with $\alpha = 0.499, \beta_1 = \beta_2 = \sqrt{\frac{1}{4} - \alpha^2}$ and $\alpha = -0.5, \beta_1 = 0, \beta_2 = 0.5$. The auxiliary variables are optimally accurate for these choices. When the remaining parameter choices from Table 3 are used, similar observations as for u_h are observed for the auxiliary variables in terms of the convergence orders, and the results are not reported here.

6.3 The third order wave equation

In this subsection, we test the third order wave equation (1.4) with the initial condition

$$u(x,0) = \sin(x), \quad \text{on} \quad (0, 2\pi), \tag{6.3}$$

and the periodic boundary condition. The exact solution is $u(x,t) = \sin(x+t)$. We test the problem with several sets of the parameters α, β_1, β_2, and compute the problem up to time $T = 5$ based on P^k approximations with $k = 1, 2, 3$. The time step is taken as $\Delta t = h$. In Table 5, we report the results for the L^2 errors and orders of accuracy of the numerical solution u_h. From these results, we see that the numerical solution u_h is optimal i.e $(k+1)$-th order with $k = 1, 2, 3$, when α, β_1, β_2 satisfy the flux conditions in (4.18). We also note that the central flux, which does not satisfies $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$, yields the suboptimal rate when k is odd. In Table 6, we present the L^2 errors and orders for the auxiliary variables q_h, p_h with $q = u_x, p = q_x$, using the parameters $\alpha = -0.499, \beta_1 = \beta_2 = -\sqrt{0.25 - 0.499^2}$, $\alpha = -0.5, \beta_1 = 0, \beta_2 = -0.5$

Table 3: L^2 errors and orders of accuracy of u_h for the fourth order equation (1.2) at time $T = 5$.

k	N	α=-0.5 β_1=β_2=0		α=0.5 β_1=β_2=0		α=-0.499 β_1=$\beta_2=\sqrt{0.25-0.499^2}$		α=-0.5 β_1=0, β_2=0.5		$\alpha=-\sqrt{0.25-0.06}$ β_1=0.2, β_2=0.3		α=0 β_1=β_2=0	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^1	10	2.20E-03	–	2.20E-03	–	2.21E-03	–	2.46E-03	–	2.32E-03	–	1.45E-03	–
	20	6.31E-04	1.80	6.31E-04	1.80	6.33E-04	1.80	6.64E-04	1.89	6.46E-04	1.85	5.84E-04	1.31
	40	1.72E-04	1.87	1.72E-04	1.87	1.72E-04	1.88	1.77E-04	1.91	1.74E-04	1.89	2.49E-04	1.23
	80	4.56E-05	1.92	4.56E-05	1.92	4.56E-05	1.92	4.64E-05	1.93	4.59E-05	1.92	1.17E-04	1.09
	160	1.18E-05	1.95	1.18E-05	1.95	1.18E-05	1.95	1.20E-05	1.96	1.19E-05	1.95	5.71E-05	1.03
	320	3.01E-06	1.97	3.01E-06	1.97	3.01E-06	1.97	3.04E-06	1.98	3.02E-06	1.97	2.84E-05	1.01
P^2	10	6.56E-04	–	6.56E-04	–	6.59E-04	–	6.98E-04	–	6.77E-04	–	6.56E-04	–
	20	8.38E-05	2.97	8.38E-05	2.97	8.39E-05	2.97	8.64E-05	3.02	8.48E-05	3.00	8.28E-05	2.99
	40	1.05E-05	3.00	1.05E-05	3.00	1.05E-05	3.00	1.06E-05	3.02	1.05E-05	3.01	1.03E-05	3.00
	80	1.31E-06	3.00	1.31E-06	3.00	1.31E-06	3.00	1.32E-06	3.01	1.30E-06	3.01	1.29E-06	3.00
	160	1.63E-07	3.00	1.63E-07	3.00	1.63E-07	3.00	1.64E-07	3.01	1.62E-07	3.01	1.61E-07	3.00
	320	2.04E-08	3.00	2.04E-08	3.00	2.04E-08	3.00	2.04E-08	3.00	2.02E-08	3.00	2.01E-08	3.00
P^3	10	1.83E-04	–	1.83E-04	–	1.84E-04	–	1.93E-04	–	1.86E-04	–	1.78E-04	–
	20	1.16E-05	3.98	1.16E-05	3.98	1.16E-05	3.98	1.20E-05	4.01	1.18E-05	3.99	1.44E-05	3.63
	40	7.26E-07	4.00	7.26E-07	4.00	7.26E-07	4.00	7.42E-07	4.01	7.31E-07	4.01	1.42E-06	3.34
	80	4.53E-08	4.00	4.53E-08	4.00	4.53E-08	4.00	4.61E-08	4.01	4.55E-08	4.01	1.64E-07	3.12
	160	2.83E-09	4.00	2.83E-09	4.00	2.83E-09	4.00	2.87E-09	4.00	2.84E-09	4.00	2.00E-08	3.03
	320	1.77E-10	4.00	1.77E-10	4.00	1.77E-10	4.00	1.79E-10	4.00	1.77E-10	4.00	2.49E-09	3.01

Table 4: L^2 errors and orders of accuracy of auxiliary variables p_h, q_h, r_h for the fourth order equation (1.2) at time $T = 5$.

k	N	α=-0.499, β_1=$\beta_2 = \sqrt{0.25 - 0.499^2}$						α=-0.5, $\beta_1 = 0$, $\beta_2 = 0.5$					
		$r - r_h$		$q - q_h$		$p - p_h$		$r - r_h$		$q - q_h$		$p - p_h$	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^1	10	2.22E-03	–	2.18E-03	–	2.59E-03	–	2.23E-03	–	2.11E-03	–	2.25E-03	–
	20	6.33E-04	1.81	6.29E-04	1.79	6.29E-04	2.04	6.30E-04	1.82	6.15E-04	1.78	6.33E-04	1.83
	40	1.72E-04	1.88	1.72E-04	1.87	1.72E-04	1.87	1.72E-04	1.87	1.71E-04	1.85	1.72E-04	1.88
	80	4.56E-05	1.92	4.56E-05	1.92	4.56E-05	1.92	4.56E-05	1.92	4.56E-05	1.90	4.56E-05	1.92
	160	1.18E-05	1.95	1.18E-05	1.95	1.18E-05	1.95	1.18E-05	1.95	1.19E-05	1.94	1.18E-05	1.95
	320	3.01E-06	1.97	3.01E-06	1.97	3.01E-06	1.97	3.01E-06	1.98	3.07E-06	1.95	3.01E-06	1.97
P^2	10	6.57E-04	–	6.44E-04	–	8.52E-04	–	6.53E-04	–	6.32E-04	–	1.03E-03	–
	20	8.39E-05	2.97	8.37E-05	2.94	8.37E-05	3.35	8.37E-05	2.96	8.37E-05	2.92	8.39E-05	3.61
	40	1.05E-05	3.00	1.05E-05	3.00	1.05E-05	3.06	1.05E-05	3.00	1.06E-05	2.98	1.05E-05	3.00
	80	1.31E-06	3.00	1.31E-06	3.00	1.31E-06	3.00	1.31E-06	3.00	1.34E-06	2.99	1.31E-06	3.00
	160	1.63E-07	3.00	1.63E-07	3.00	1.63E-07	3.00	1.63E-07	3.00	1.68E-07	3.00	1.63E-07	3.00
	320	2.04E-08	3.00	2.04E-08	3.00	2.04E-08	3.00	2.04E-08	3.00	2.10E-08	3.00	2.04E-08	3.00
P^3	10	1.84E-04	–	1.84E-04	–	2.64E-04	–	1.83E-04	–	1.81E-04	–	2.64E-04	–
	20	1.16E-05	3.98	1.16E-05	3.99	1.16E-05	4.51	1.16E-05	3.98	1.15E-05	3.97	1.16E-05	4.50
	40	7.26E-07	4.00	7.26E-07	4.00	7.26E-07	4.02	7.26E-07	4.00	7.28E-07	3.99	7.26E-07	4.00
	80	4.53E-08	4.00	4.53E-08	4.00	4.53E-08	4.00	4.53E-08	4.00	4.57E-08	4.00	4.53E-08	4.00
	160	2.83E-09	4.00	2.83E-09	4.00	2.83E-09	4.00	2.83E-09	4.00	2.86E-09	4.00	2.83E-09	4.00
	320	1.77E-10	4.00	1.77E-10	4.00	1.77E-10	4.00	1.77E-10	4.00	1.79E-10	4.00	1.77E-10	4.00

and $\alpha = \beta_1 = \beta_2 = 0$. We observe that the auxiliary variables p_h, q_h have the same accuracy as u_h, with one exception when the central fluxes are used for u_h and p_h, and $F_q = q_h^+$ for q_h. In this case, q_h is always optimal, while u_h and p_h are suboptimal when k is odd.

6.4 The linear Schrödinger equation

Finally, we test the one dimensional linear Schrödinger problem (1.5) with the initial data

$$u(x, 0) = \sin(x), \quad \text{on} \quad (0, 2\pi), \tag{6.4}$$

and with the periodic boundary condition. The exact solution is $u(x,t) = e^{i(x-t)}$. We test this problem with different sets of α, β_1, β_2, and compute up to $T = 5$ based on P^k approximations with $k = 1, 2, 3$. In our test for this problem, the CFL constraint for the method with central flux ($\alpha = \beta_1 = \beta_2 = 0$) is taken to be 0.2 and 1.0 for other choices of parameters in numerical fluxes. We show the numerical results in Table 7 and Table 8 for L^2 errors and orders of accuracy of the numerical solutions u_h and the auxiliary variable p_h. From these results, we see both the numerical solution u_h and auxiliary variable p_h have the optimal accuracy with $k = 1, 2, 3$ under the conditions $\alpha^2 - \beta_1\beta_2 = \frac{1}{4}, \beta_1 \leq 0, \beta_2 \geq 0$. On the other hand, the DG method with the central flux (with $\alpha = \beta_1 = \beta_2 = 0$) yields the suboptimal accuracy.

7 Conclusion

In this paper, we have developed DG methods for solving the even-order equations (including the heat and a fourth order equation), a third order wave equation, and the linear Schrödinger equation in one dimension. A general class of numerical fluxes is identified to ensure the optimal accuracy of the numerical solution and of some auxiliary variables. A set of energy relations, as well as the design of special projection operators are the key to achieve the optimality of the error estimates. In future work, we want to extend the study to high dimensions and to nonlinear models.

A The proof of Theorem 4.3

In this appendix, we prove the L^2 stability of the DG scheme (4.6)-(4.7) with the numerical fluxes (4.5) and $\alpha = \frac{1}{2}, \beta_1 = \beta_2 = 0$, namely

$$F_p = p_h^+, \quad F_q = q_h^+, \quad F_u = u_h^-. \tag{A.1}$$

Table 5: L^2 errors and orders of accuracy of u_h for the third order wave equation (1.4) at time $T = 5$.

k	N	α=-0.5, β_1=β_2=0		α=0.5, β_1=β_2=0		α=-0.499, β_1=$\beta_2 = -\sqrt{0.25-0.499^2}$		α=-0.5, β_1=0, β_2=-0.5		α=$-\sqrt{0.25-0.02}$, β_1=-0.1, β_2=-0.2		α=0, β_1=β_2=0	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^1	10	1.78E-01	–	1.92E-01	–	1.78E-01	–	1.87E-01	–	1.80E-01	–	2.68E-01	–
	20	6.10E-02	1.55	6.69E-02	1.52	6.10E-02	1.55	6.20E-02	1.59	6.09E-02	1.56	1.14E-01	1.23
	40	1.88E-02	1.70	1.96E-02	1.77	1.88E-02	1.70	1.92E-02	1.69	1.88E-02	1.69	4.93E-02	1.22
	80	5.15E-03	1.87	5.25E-03	1.90	5.15E-03	1.87	5.30E-03	1.86	5.18E-03	1.86	2.30E-02	1.10
	160	1.34E-03	1.94	1.36E-03	1.95	1.34E-03	1.94	1.39E-03	1.93	1.35E-03	1.94	1.13E-02	1.03
	320	3.42E-04	1.97	3.44E-04	1.98	3.43E-04	1.97	3.55E-04	1.97	3.46E-04	1.97	5.62E-03	1.01
P^2	10	9.82E-02	–	9.72E-02	–	9.84E-02	–	1.03E-01	–	9.95E-02	–	9.78E-02	–
	20	1.27E-02	2.95	1.25E-02	2.96	1.27E-02	2.95	1.33E-02	2.95	1.29E-02	2.95	1.25E-02	2.97
	40	1.58E-03	3.01	1.56E-03	3.00	1.58E-03	3.01	1.65E-03	3.01	1.60E-03	3.01	1.55E-03	3.01
	80	1.96E-04	3.01	1.94E-04	3.01	1.96E-04	3.01	2.04E-04	3.01	1.98E-04	3.01	1.92E-04	3.01
	160	2.43E-05	3.01	2.42E-05	3.00	2.44E-05	3.01	2.54E-05	3.01	2.46E-05	3.01	2.39E-05	3.01
	320	3.03E-06	3.00	3.06E-06	3.00	3.04E-06	3.00	3.16E-06	3.00	3.07E-06	3.00	2.99E-06	3.00
P^3	10	2.70E-02	–	2.70E-02	–	2.70E-02	–	2.79E-02	–	2.72E-02	–	2.72E-02	–
	20	1.71E-03	3.98	1.72E-03	3.98	1.72E-03	3.98	1.77E-03	3.98	1.73E-03	3.98	2.28E-03	3.58
	40	1.08E-04	3.99	1.08E-04	3.99	1.08E-04	3.99	1.12E-04	3.99	1.10E-04	3.99	2.35E-04	3.28
	80	6.78E-06	3.99	6.78E-06	3.99	6.79E-06	3.99	7.04E-06	3.99	6.85E-06	3.99	2.76E-05	3.09
	160	4.25E-07	4.00	4.25E-07	4.00	4.25E-07	4.00	4.40E-07	4.00	4.28E-07	4.00	3.39E-06	3.02
	320	2.66E-08	4.00	2.66E-08	4.00	2.66E-08	4.00	2.75E-08	4.00	2.68E-08	4.00	4.22E-07	3.01

Table 6: L^2 errors and orders of accuracy of auxiliary variables p_h, q_h for the third order wave equation (1.4) at time $T = 5$.

k	N	α=-0.499,β_1=$\beta_2 = -\sqrt{0.25 - 0.499^2}$				α=-0.5,β_1=0, β_2=-0.5				α=0, $\beta_1 = \beta_2 = 0$			
		$q - q_h$		$p - p_h$		$q - q_h$		$p - p_h$		$q - q_h$		$p - p_h$	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^1	10	1.77E-01	–	1.93E-01	–	1.78E-01	–	1.95E-01	–	1.98E-01	–	2.31E-01	–
	20	6.11E-02	1.55	6.70E-02	1.52	6.10E-02	1.54	6.73E-02	1.59	7.30E-02	1.44	9.43E-02	1.30
	40	1.88E-02	1.70	1.96E-02	1.77	1.87E-02	1.70	1.96E-02	1.78	2.19E-02	1.74	3.65E-02	1.37
	80	5.15E-03	1.87	5.25E-03	1.90	5.14E-03	1.87	5.25E-03	1.90	5.90E-03	1.89	1.59E-02	1.20
	160	1.34E-03	1.94	1.35E-03	1.95	1.34E-03	1.94	1.36E-03	1.95	1.53E-03	1.95	7.61E-03	1.07
	320	3.43E-04	1.97	3.44E-04	1.98	3.42E-04	1.97	3.44E-04	1.98	3.88E-04	1.98	3.75E-03	1.02
P^2	10	9.88E-02	–	9.70E-02	–	9.88E-02	–	9.73E-01	–	9.86E-02	–	9.79E-02	–
	20	1.27E-02	2.96	1.25E-02	2.96	1.27E-02	2.96	1.25E-02	2.96	1.27E-02	2.95	1.25E-02	2.97
	40	1.58E-03	3.01	1.56E-03	3.00	1.58E-03	3.01	1.56E-03	3.00	1.58E-03	3.01	1.55E-03	3.01
	80	1.95E-04	3.01	1.94E-04	3.01	1.96E-04	3.01	1.94E-04	3.01	1.96E-04	3.01	1.92E-04	3.01
	160	2.43E-05	3.01	2.42E-05	3.00	2.43E-05	3.01	2.42E-05	3.00	2.43E-05	3.01	2.39E-05	3.01
	320	3.03E-06	3.00	3.02E-06	3.00	3.03E-06	3.00	3.03E-06	3.00	3.03E-06	3.00	2.99E-06	3.00
P^3	10	2.70E-02	–	2.83E-02	–	2.71E-02	–	2.71E-02	–	2.68E-02	–	2.61E-02	–
	20	1.71E-03	3.98	1.71E-03	4.04	1.71E-03	3.98	1.71E-03	3.98	1.72E-03	3.98	2.11E-03	3.62
	40	1.08E-04	3.99	1.08E-04	3.99	1.08E-04	3.99	1.08E-04	3.99	1.08E-04	3.99	2.08E-04	3.35
	80	6.78E-06	3.99	6.77E-06	3.99	6.78E-06	3.99	6.78E-06	3.99	6.74E-06	3.99	2.39E-05	3.12
	160	4.25E-07	4.00	4.24E-07	4.00	4.25E-07	4.00	4.25E-07	4.00	4.22E-07	4.00	2.91E-06	3.03
	320	2.66E-08	4.00	2.65E-08	4.00	2.66E-08	4.00	2.66E-08	4.00	2.64E-08	4.00	3.62E-07	3.01

Table 7: L^2 errors and orders of accuracy for u_h for the linear Schrödinger equation (1.5) at $T = 5$.

k	N	α=-0.5 β_1=β_2=0		α=0.5 β_1=β_2=0		α=-0.499 $-\beta_1$=$\beta_2 = \sqrt{0.25-0.499^2}$		α=-0.5 β_1=-0.5, β_2=0		α=$-\sqrt{0.25-0.06}$ β_1=-0.2, β_2=0.3		α=0.5 β_1=0., β_2=0.5		α=0 β_1=β_2=0	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^1	10	2.72E-01	–	2.49E-01	–	2.70E-01	–	2.70E-01	–	2.57E-01	–	2.54E-01	–	3.63E-01	–
	20	9.51E-02	1.52	8.66E-02	1.52	9.45E-02	1.52	9.46E-02	1.52	8.98E-02	1.52	8.85E-02	1.52	1.32E-01	1.46
	40	2.78E-02	1.77	2.66E-02	1.70	2.77E-02	1.77	2.77E-02	1.77	2.69E-02	1.74	2.67E-02	1.73	8.02E-02	0.72
	80	7.44E-03	1.90	7.29E-03	1.87	7.42E-03	1.90	7.43E-03	1.90	7.28E-03	1.89	7.25E-03	1.88	4.34E-02	0.89
	160	1.92E-03	1.96	1.90E-03	1.94	1.92E-03	1.95	1.92E-03	1.95	1.89E-03	1.95	1.88E-03	1.94	2.18E-02	1.00
	320	4.87E-04	1.98	4.85E-04	1.97	4.86E-04	1.98	4.87E-04	1.98	4.80E-04	1.98	4.79E-04	1.97	1.09E-02	1.00
P^2	10	1.38E-01	–	1.40E-01	–	1.37E-01	–	1.38E-01	–	1.37E-01	–	1.37E-01	–	1.38E-01	–
	20	1.77E-02	2.96	1.80E-02	2.96	1.77E-02	2.96	1.77E-02	2.96	1.76E-02	2.96	1.76E-02	2.96	1.76E-02	2.97
	40	2.21E-03	3.00	2.23E-03	3.01	2.21E-03	3.00	2.21E-03	3.00	2.19E-03	3.01	2.19E-03	3.01	2.19E-03	3.01
	80	2.75E-04	3.01	2.77E-04	3.01	2.75E-04	3.01	2.75E-04	3.01	2.72E-04	3.01	2.72E-04	3.01	2.72E-04	3.01
	160	3.43E-05	3.00	3.44E-05	3.01	3.42E-05	3.00	3.43E-05	3.00	3.39E-05	3.00	3.39E-05	3.01	3.38E-05	3.01
	320	4.28E-06	3.00	4.29E-06	3.00	4.27E-06	3.00	4.38E-06	3.00	4.23E-06	3.00	4.22E-06	3.00	4.22E-06	3.00
P^3	10	3.82E-02	–	3.83E-02	–	3.82E-02	–	3.83E-02	–	3.78E-02	–	3.78E-02	–	3.79E-02	–
	20	2.43E-03	3.98	2.42E-03	3.98	2.42E-03	3.98	2.43E-03	3.98	2.40E-03	3.98	2.40E-03	3.98	3.03E-03	3.64
	40	1.53E-04	3.99	1.53E-04	3.99	1.53E-04	3.99	1.53E-04	3.99	1.51E-04	3.99	1.51E-04	3.99	2.08E-04	3.86
	80	9.59E-06	3.99	9.59E-06	3.99	9.58E-06	3.99	9.59E-06	3.99	9.48E-06	3.99	9.47E-06	3.99	1.92E-05	3.44
	160	6.01E-07	4.00	6.00E-07	4.00	6.00E-07	4.00	6.01E-07	4.00	5.94E-07	4.00	5.93E-07	4.00	4.17E-06	2.20
	320	3.76E-08	4.00	3.76E-08	4.00	3.75E-08	4.00	3.76E-08	4.00	3.71E-08	4.00	3.72E-08	4.00	5.86E-07	2.83

Table 8: L^2 errors and orders of accuracy for p_h with $p = u_x$ for the linear Schrödinger equation (1.5) at $T = 5$.

k	N	α=-0.5 β_1=β_2=0		α=0.5 β_1=β_2=0		α=-0.499 $-\beta_1$=$\beta_2 = \sqrt{0.25-0.499^2}$		α=-0.5 β_1=-0.5, β_2=0		α=$-\sqrt{0.25-0.06}$ β_1=-0.2, β_2=0.3		α=0.5 β_1=0, β_2=0.5		α=0 β_1=β_2=0	
		L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order	L^2 error	order
P^1	10	2.49E-01	–	2.73E-01	–	2.49E-01	–	2.62E-01	–	2.52E-01	–	2.50E-01	–	3.68E-01	–
	20	8.63E-02	1.53	9.54E-02	1.52	8.63E-02	1.53	8.77E-02	1.58	8.62E-02	1.55	8.62E-02	1.54	1.14E-01	1.69
	40	2.66E-02	1.70	2.79E-02	1.77	2.66E-02	1.70	2.72E-02	1.69	2.66E-02	1.69	2.65E-02	1.70	7.42E-02	0.62
	80	7.29E-03	1.87	7.45E-03	1.90	7.30E-03	1.87	7.51E-03	1.86	7.32E-03	1.86	7.28E-03	1.87	5.38E-02	0.47
	160	1.90E-03	1.94	1.92E-03	1.96	1.90E-03	1.94	1.96E-03	1.93	1.91E-03	1.94	1.90E-03	1.94	2.87E-02	0.90
	320	4.85E-04	1.97	4.87E-04	1.98	4.85E-04	1.97	5.02E-04	1.97	4.88E-04	1.97	4.84E-04	1.97	1.45E-02	0.98
P^2	10	1.40E-01	–	1.38E-01	–	1.40E-01	–	1.46E-01	–	1.41E-01	–	1.39E-01	–	1.38E-01	–
	20	1.80E-02	2.96	1.77E-02	2.96	1.80E-02	2.96	1.88E-02	2.95	1.82E-02	2.95	1.80E-02	2.95	1.76E-02	2.97
	40	2.23E-03	3.01	2.21E-03	3.00	2.24E-03	3.01	2.33E-03	3.01	2.25E-03	3.01	2.23E-03	3.01	2.19E-03	3.01
	80	2.77E-04	3.01	2.75E-04	3.01	2.77E-04	3.01	2.89E-04	3.01	2.79E-04	3.01	2.77E-04	3.01	2.72E-04	3.01
	160	3.44E-05	3.01	3.43E-05	3.00	3.45E-05	3.01	3.59E-05	3.01	3.47E-05	3.01	3.44E-05	3.01	3.38E-05	3.01
	320	4.29E-06	3.00	4.28E-06	3.00	4.29E-06	3.00	4.47E-06	3.00	4.29E-06	3.00	4.29E-06	3.00	4.22E-06	3.00
P^3	10	3.82E-02	–	3.82E-02	–	3.83E-02	–	3.96E-02	–	3.85E-02	–	3.82E-02	–	3.79E-02	–
	20	2.42E-03	3.98	2.43E-03	3.98	2.43E-03	3.98	2.51E-03	3.98	2.44E-03	3.98	2.42E-03	3.98	3.03E-03	3.64
	40	1.53E-04	3.99	1.53E-04	3.99	1.53E-04	3.99	1.58E-04	3.99	1.54E-04	3.99	1.53E-04	3.99	1.76E-03	0.78
	80	9.59E-06	3.99	9.59E-06	3.99	9.60E-06	3.99	9.93E-06	3.99	9.66E-06	3.99	9.59E-06	3.99	2.88E-04	2.62
	160	6.00E-07	4.00	6.01E-07	4.00	6.01E-07	4.00	6.22E-07	4.00	6.05E-07	4.00	6.00E-07	4.00	3.17E-05	3.18
	320	3.76E-08	4.00	3.76E-08	4.00	3.76E-08	4.00	3.89E-08	4.00	3.78E-08	4.00	3.76E-08	4.00	3.87E-06	3.03

Five energy equations will be derived first.

• **The first energy equation.** With $\alpha = \frac{1}{2}, \beta_1 = \beta_2 = 0$, the first energy equation (4.9) becomes

$$0 = B(u_h, p_h, q_h; u_h, q_h, -p_h) = \frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx + \frac{1}{2}\sum_j [q_h]^2_{j-\frac{1}{2}}. \tag{A.2}$$

• **The second energy equation.** We take the test functions $v = -\frac{1}{2}(q_h)_t$, $w = \frac{1}{2}p_h$ and $z = 0$ in (4.10), use the definition of F_p, F_q and F_u in (A.1), and obtain

$$\begin{aligned} 0 =& B(u_h, (p_h)_t, (q_h)_t; -\frac{1}{2}(q_h)_t, \frac{1}{2}p_h, 0) \\ =& -\frac{1}{2}\int_\Omega (u_h)_t(q_h)_t dx + \frac{1}{2}\sum_j \left(\int_{I_j} p_h (q_h)_{tx} dx + (F_p[(q_h)_t])_{j-\frac{1}{2}}\right) \\ &+ \frac{1}{2}\int_\Omega (p_h)_t p_h dx + \frac{1}{2}\sum_j \left(\int_{I_j} (q_h)_t (p_h)_x dx + ((F_q)_t[p_h])_{j-\frac{1}{2}}\right) \\ =& \frac{1}{2}\int_\Omega (-(u_h)_t(q_h)_t + (p_h)_t p_h) dx + \frac{1}{2}\sum_j ([p_h][(q_h)_t])_{j-\frac{1}{2}}. \end{aligned} \tag{A.3}$$

• **The third energy equation.** With $\alpha = \frac{1}{2}, \beta_1 = \beta_2 = 0$, (4.14) becomes

$$0 = B((u_h)_t, (p_h)_t, (q_h)_t; (u_h)_t, (q_h)_t, -(p_h)_t) = \frac{1}{2}\frac{d}{dt}\int_\Omega (u_h)_t^2 dx + \frac{1}{2}\sum_j [(q_h)_t]^2_{j-\frac{1}{2}}. \tag{A.4}$$

• **The fourth energy equation.** Here we take the test functions $v = -p_h$, $w = (u_h)_t$ and $z = q_h$ in (4.15), use the definition of F_p, F_q and F_u in (A.1), and obtain

$$\begin{aligned} 0 =& B(u_h, p_h, (q_h)_t; -p_h, (u_h)_t, q_h) \\ =& -\int_\Omega (u_h)_t p_h dx + \sum_j (\int_{I_j} p_h (p_h)_x dx + (F_p[p_h])_{j-\frac{1}{2}}) \\ &+ \int_\Omega p_h (u_h)_t dx + \sum_j (\int_{I_j} q_h (u_h)_{tx} dx + (F_q[(u_h)_t])_{j-\frac{1}{2}}) \\ &+ \int_\Omega (q_h)_t q_h dx + \sum_j (\int_{I_j} (u_h)_t (q_h)_x dx + ((F_u)_t[q_h])_{j-\frac{1}{2}}) \\ =& \frac{1}{2}\frac{d}{dt}\int_\Omega (q_h)^2 dx + \frac{1}{2}\sum_j [p_h]^2_{j-\frac{1}{2}}. \end{aligned} \tag{A.5}$$

• **The fifth energy equation.** We take the time derivative of (4.15) and get

$$B((u_h)_t,(p_h)_t,(q_h)_{tt};v,w,z)=0,\quad \forall v,w,z\in\mathcal{V}_h^k. \tag{A.6}$$

With the test functions in (A.6) taken as $v=-(p_h)_t$, $w=(u_h)_{tt}$ and $z=(q_h)_t$, using the definition of F_p, F_q and F_u in (A.1), we have

$$0=B((u_h)_t,(p_h)_t,(q_h)_{tt};-(p_h)_t,(u_h)_{tt},(q_h)_t)=\int_\Omega (q_h)_{tt}(q_h)_t dx+\frac{1}{2}\sum_j [(p_h)_t]^2_{j-\frac{1}{2}}. \tag{A.7}$$

• **Proof of Theorem 4.3**

Proof. We sum up the energy equations (A.2)-(A.5) and (A.7), and get

$$\begin{aligned}0=&\frac{1}{2}\frac{d}{dt}\left(\|u_h\|_\Omega^2+\frac{1}{2}\|p_h\|_\Omega^2+\|(u_h)_t\|_\Omega^2+\|q_h\|_\Omega^2+\|(q_h)_t\|_\Omega^2\right)-\frac{1}{2}((q_h)_t,(u_h)_t)\\&+\frac{1}{2}\sum_j\left([q_h]^2+[(q_h)_t]^2+[(p_h)_t]^2+[p_h]^2+[p_h][(q_h)_t]\right)_{j-\frac{1}{2}}.\end{aligned} \tag{A.8}$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u_h\|_\Omega^2+\frac{1}{2}\|p_h\|_\Omega^2+\|(u_h)_t\|_\Omega^2+\|q_h\|_\Omega^2+\|(q_h)_t\|_\Omega^2\right)\le\frac{1}{4}(\|(q_h)_t\|_\Omega^2+\|(u_h)_t\|_\Omega^2). \tag{A.9}$$

We integrate (A.9) with respect to time over $[0,T]$, and obtain

$$\begin{aligned}&\|u_h(T)\|_\Omega^2+\frac{1}{2}\|p_h(T)\|_\Omega^2+\|(u_h)_t(T)\|_\Omega^2+\|q_h(T)\|_\Omega^2+\|(q_h)_t(T)\|_\Omega^2\\&\le\frac{1}{2}\int_0^T(\|(q_h)_t(t)\|_\Omega^2+\|(u_h)_t(t)\|_\Omega^2)dt\\&+\left(\|u_h(0)\|_\Omega^2+\frac{1}{2}\|p_h(0)\|_\Omega^2+\|(u_h)_t(0)\|_\Omega^2+\|q_h(0)\|_\Omega^2+\|(q_h)_t(0)\|_\Omega^2\right).\end{aligned} \tag{A.10}$$

From the third energy equation (A.4) and the fifth energy equation (A.7), we have

$$\|(u_h)_t(t)\|_\Omega^2\le\|(u_h)_t(0)\|_\Omega^2,\quad \|(q_h)_t(t)\|_\Omega^2\le\|(q_h)_t(0)\|_\Omega^2,\quad\forall t\ge 0. \tag{A.11}$$

Therefore, we can obtain the L^2 stability in Theorem 4.3. □

B The derivation of the conditions (4.18)

In this appendix, we will give the derivation of the conditions (4.18) that are to ensure the matrices S_1 and S_2 in (4.22) to be positive semi-definite. For this, we use the following sufficient and necessary condition for an $n \times n$ matrix to be positive semi-definite: *all the principal minors D_k are nonnegative, $k = 1, \cdots n$*. Here, D_k is formed by deleting *any* $n-k$ rows and the corresponding columns. Additionally, we require the relation $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$, which helps with simplifying the conditions and is also needed for optimal accuracy.

For the matrix S_1, the first principal minors are

$$D_{1,1} = -\beta_1,\ D_{1,2} = -\beta_2,\ D_{1,3} = \frac{1}{2}; \tag{B.1}$$

the second principal minors are

$$D_{2,1} = \begin{vmatrix} -\beta_2 & \frac{\alpha+0.5}{2} \\ \frac{\alpha+0.5}{2} & \frac{1}{2} \end{vmatrix},\ D_{2,2} = \begin{vmatrix} -\beta_1 & \frac{\beta_1}{2} \\ \frac{\beta_1}{2} & \frac{1}{2} \end{vmatrix},\ D_{2,3} = \begin{vmatrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{vmatrix}; \tag{B.2}$$

and the third principal minor is

$$D_3 = |S_1| = \frac{1}{2}\beta_1\beta_2 + \frac{1}{4}\beta_1^2\beta_2 + \frac{1}{4}\beta_1(\alpha + \frac{1}{2})^2. \tag{B.3}$$

Let the first principal minors be nonnegative, we have $\beta_1 \leq 0$ and $\beta_2 \leq 0$. From the second principal minors being nonnegative, we obtain

$$2\beta_2 + (\alpha + \frac{1}{2})^2 \leq 0,\ \ \beta_1(2+\beta_1) \leq 0,\ \ \beta_1\beta_2 \geq 0. \tag{B.4}$$

Let the third principal minor D_3 be nonnegative, with $\beta_1 \leq 0$ and the assumption $\alpha^2 + \beta_1\beta_2 = \frac{1}{4}$, we get

$$2\beta_2 + \beta_1\beta_2 + (\alpha + \frac{1}{2})^2 = 2\beta_2 + \alpha + \frac{1}{2} \leq 0. \tag{B.5}$$

We observe that, with $\beta_1\beta_2 \geq 0$, the inequality (B.5) will automatically ensure the first inequality in (B.4). Combining all we have so far, the following conditions are derived to ensure S_1 be positive semi-definite

$$-2 \leq \beta_1 \leq 0,\ \ \beta_2 \leq 0,\ \ 2\beta_2 + \alpha \leq -\frac{1}{2},\ \ \alpha^2 + \beta_1\beta_2 = \frac{1}{4}. \tag{B.6}$$

For the matrix S_2, we follow the similar analysis as for S_1. By requiring all the first and the second principal minors being nonnegative, we get

$$\beta_1 + \alpha \leq 0,\quad \beta_2 \leq 0, \tag{B.7}$$

$$8(\beta_1+\alpha)+(\frac{1}{2}-\alpha)^2 \le 0, \quad \beta_2(8+\beta_2)\le 0, \quad (\alpha+\beta_1)\beta_2-\frac{1}{4}\beta_2^2 \ge 0, \tag{B.8}$$

Let the third order principal minor of S_2 be nonnegative, also with $\beta_2 \ge 0$, we obtain

$$\begin{aligned} 8(\beta_1+\alpha)+&(\frac{1}{2}-\alpha)^2+\beta_2(\frac{1}{2}-\alpha)+\beta_2(\beta_1+\alpha)-2\beta_2 \\ &= 8(\beta_1+\alpha)+(\frac{1}{2}-\alpha)^2+\beta_2(\beta_1-\frac{3}{2}) \le 0. \end{aligned} \tag{B.9}$$

Using $\beta_2 \le 0$ and assuming $\beta_1 \le 0$, one can see that (B.9) implies the first inequality in (B.8), which on the other hand ensures the first inequality in (B.7). Combining (B.7)-(B.9) with $\alpha^2+\beta_1\beta_2=\frac{1}{4}$, we have the conditions for S_2 as

$$\alpha^2+\beta_1\beta_2=\frac{1}{4}, \ \beta_1\le 0, \ -8\le\beta_2\le 0, \ 4(\beta_1+\alpha)\le\beta_2, \ 8\beta_1+7\alpha-\frac{3}{2}\beta_2\le-\frac{1}{2}. \tag{B.10}$$

Finally, we reach the conditions in (4.18) by putting (B.6) and (B.10) together. To get some idea about these conditions in (4.18), we present two plots in Figure 1. In the left figure, we plot those pairs (β_1,β_2) such that with the respective $\alpha=-\sqrt{\frac{1}{4}-\beta_1\beta_2}$, the conditions in (4.18) are all satisfied. In the right figure, we fix $\alpha=-\frac{1}{4}$, and plot (β_1,β_2) satisfying the conditions in (4.18). Note that such (β_1,β_2) form part of the parabola: $\beta_1\beta_2=\frac{1}{4}-\alpha^2=\frac{3}{16}$, see the solid line in red.

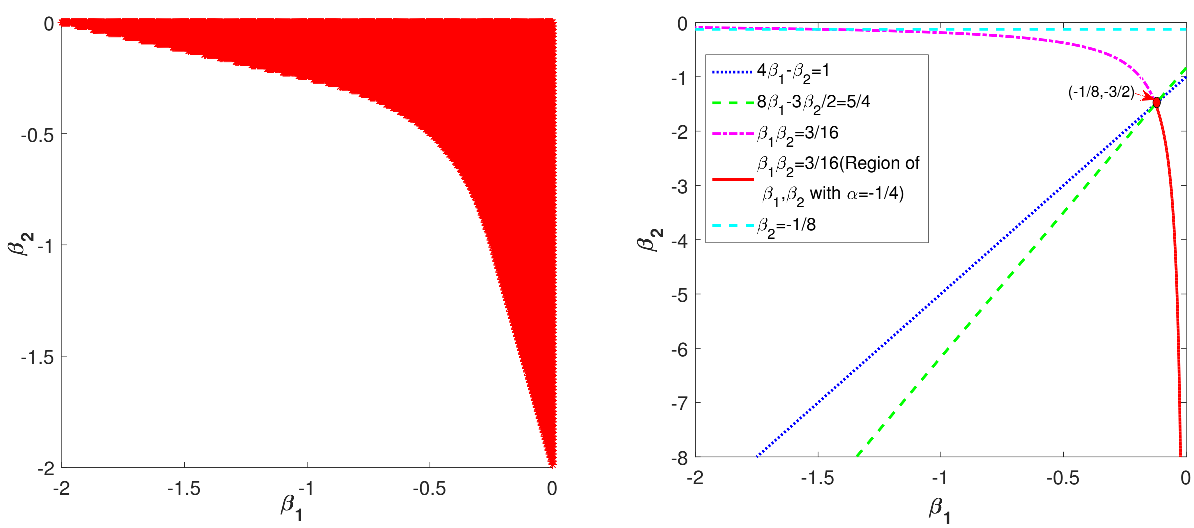

Figure 1: The region of β_1,β_2 in the condition (4.18) for the stability. Left: $\alpha=-\sqrt{\frac{1}{4}-\beta_1\beta_2}$; Right: $\alpha=-\frac{1}{4}$.

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