A priori error estimates of a local-structure-preserving LDG method

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Abstract: *In this note, the missing error estimates in the L^2 norm and in a new energy norm are established for a local-structure-preserving local discontinuous Galerkin method proposed in [F. Li and C.-W. Shu, Methods and Applications of Analysis, v13 (2006), pp.215-233] for the Laplace equation. With its distinctive feature in using harmonic polynomials as local approximations, the method has lower computational complexity than standard discontinuous Galerkin methods. The analysis in this note is based on the primal formulation of the scheme.*

1 Introduction

The primary goal of this note is to establish the missing L^2 error estimate for a local-structure-preserving local discontinuous Galerkin (LDG) method in [6] for solving the Laplace equation. The method is based on the standard LDG method for elliptic equations ([2]), and its distinctive feature of the method is the use of harmonic polynomials (polynomials which satisfy $\triangle u = 0$) as local approximations for the solution u. That is, the numerical solution satisfies the differential equation exactly in each mesh element. Using such a local-structure-preserving approximation space significantly reduces the size of the matrix in the final system and therefore the overall computational complexity.

In [6], error estimates in the energy norm were established for the local-structure-preserving LDG (LSP-LDG) method, and they are confirmed numerically. Numerical experiments also suggest an optimal L^2 error estimate. In this note, this missing L^2 error estimate will be proved. Though one can still work with the original flux formulation of the bilinear form of the scheme as in [6], the proof in this note is instead based on the primal formulation ([1]), and this leads to a more concise analysis for the L^2 error estimate. In particular, the scheme in its primal formulation is bounded and coercive with respect to a new energy norm $|\cdot|_{\mathcal{B}}$ (defined in (3.6)), and it is also consistent and adjoint consistent. Together with the approximation properties of the locally harmonic polynomial functions for the solution u of the Laplace equation, the optimal error estimates in both the L^2 norm and the new energy norm $|\cdot|_{\mathcal{B}}$ will come naturally.

The analysis in this note is closely related to the analysis in [1] which is to study the standard LDG method in its primal formulation. The main difference is in the definition of energy norms. With the new energy norm $||\cdot||_{\mathcal{B}}$, which is different from the one in [6] (defined below by (2.12)) and is often used to analyze the (symmetric) interior penalty method ([5]), one avoids many technical complications which would otherwise be needed. Moreover, the analysis in this note provides a more straightforward understanding than [6] for the role of the approximating space of the auxiliary variable $\mathbf{q} = \nabla u$ in terms of the accuracy of the LSP-LDG method. The rest of the note is organized as follows. In Section 2, both the standard and LSP-LDG methods are reviewed for solving the Laplace equation. The optimal error estimates in the L^2 norm as well as in a new energy norm $|\cdot|_{\mathcal{B}}$ are established in Section 3.

2 Review of the methods

In this section, the standard ([2]) and the local-structure-preserving LDG ([6]) methods will be reviewed for the Laplace equation

$$-\triangle u = 0 \quad \text{in } \Omega, \qquad u|_{\Gamma_D} = g_D, \qquad \frac{\partial u}{\partial \mathbf{n}}|_{\Gamma_N} = \mathbf{g}_N \cdot \mathbf{n}, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with $\mathbf{n}$ as the unit outer normal along the domain boundary $\Gamma = \Gamma_N \cup \Gamma_D$, and $|\Gamma_D|_{\mathbb{R}^{d-1}} > 0$. Though numerical methods presented below can be formulated for general space dimension, they are discussed and analyzed only for $d = 2$ in this note.

We start with a partition $\mathcal{T}_h = \{K\}$ of the domain Ω, with the triangular or rectangular element being denoted as K, the edge as e, the diameter of K as h_K, and the meshsize of $\mathcal{T}_h$ as $h = \max_{K\in\mathcal{T}_h} h_K$. We further denote the union of all interior edges as $\mathcal{E}_i$, the union of boundary edges in Γ_D (resp. Γ_N) as $\mathcal{E}_D$ (resp. $\mathcal{E}_N$), and $\mathcal{E} = \mathcal{E}_i \cup \mathcal{E}_D \cup \mathcal{E}_N$. With an auxiliary variable $\mathbf{q} = \nabla u$, (2.1) can be rewritten as

$$\mathbf{q} = \nabla u, \qquad -\nabla \cdot \mathbf{q} = 0 \ \text{ in } \Omega, \qquad u|_{\Gamma_D} = g_D, \qquad \mathbf{q}\cdot\mathbf{n}|_{\Gamma_N} = \mathbf{g}_N \cdot \mathbf{n}. \tag{2.2}$$

Based on [2], the LDG method for solving (2.2) can be formulated as: finding $(u_h, \mathbf{q}_h) \in (V_h, \mathbf{M}_h)$, such that

$$\int_K \mathbf{q}_h \cdot \mathbf{r} dx = -\int_K u_h \nabla \cdot \mathbf{r} dx + \int_{\partial K} \hat{u}_h \mathbf{r} \cdot \mathbf{n}_K ds, \tag{2.3}$$

$$\int_K \mathbf{q}_h \cdot \nabla v dx = \int_{\partial K} v \hat{\mathbf{q}}_h \cdot \mathbf{n}_K ds \tag{2.4}$$

for all $(v, \mathbf{r}) \in (V_h, \mathbf{M}_h)$ and $K \in \mathcal{T}_h$. Here $\mathbf{n}_K$ is the unit outer normal of K, $(V_h, \mathbf{M}_h)$ is the discrete space pair to approximate $(u, \mathbf{q})$, and $(\hat{u}_h, \hat{\mathbf{q}}_h)$ are the so-called numerical fluxes, which are single-valued functions approximating the trace of $(u, \mathbf{q})$ on $\mathcal{E}$. To finalize the scheme, one needs to specify the definitions of $(\hat{u}_h, \hat{\mathbf{q}}_h)$ and $(V_h, \mathbf{M}_h)$.

The LSP-LDG method in [6] uses the same numerical fluxes defined in [2] for the standard LDG method, namely, for the interior edge $e \in \mathcal{E}_i$,

$$\hat{\mathbf{q}}_h = \{\{\mathbf{q}_h\}\} - C_{11}[\![u_h]\!] - \mathbf{C}_{12}[\![\mathbf{q}_h]\!], \qquad \hat{u}_h = \{\{u_h\}\} + \mathbf{C}_{12} \cdot [\![u_h]\!], \tag{2.5}$$

and on the boundary,

$$\hat{\mathbf{q}}_h = \begin{cases} \mathbf{q}_h^+ - C_{11}(u_h^+ - g_D)\mathbf{n} & \text{for } e \in \mathcal{E}_D, \\ \mathbf{g}_N & \text{for } e \in \mathcal{E}_N, \end{cases} \qquad \hat{u}_h = \begin{cases} g_D & \text{for } e \in \mathcal{E}_D, \\ u_h^+ & \text{for } e \in \mathcal{E}_N. \end{cases} \tag{2.6}$$

Here the standard notations are used for the average $\{\{\cdot\}\}$ and the jump $[\![\cdot]\!]$. Suppose $e \in \mathcal{E}_i$ and $e = K^+ \cap K^-$, $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) is the unit outer normal of K^+ (resp. K^-), and $(v^+, \mathbf{r}^+)$ (resp. $(v^-, \mathbf{r}^-)$) is the trace of the piecewise smooth function $(v, \mathbf{r})$ from the interior of K^+ (resp. K^-) along e. Then on this edge e,

$$\{\{v\}\} = (v^+ + v^-)/2, \qquad \{\{\mathbf{r}\}\} = (\mathbf{r}^+ + \mathbf{r}^-)/2, \tag{2.7}$$

$$[\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^-, \qquad [\![\mathbf{r}]\!] = \mathbf{r}^+ \cdot \mathbf{n}^+ + \mathbf{r}^- \cdot \mathbf{n}^-. \tag{2.8}$$

In addition, for $e \in \mathcal{E}_D$ and $e \subset \partial K$, we define $[\![v]\!] = v|_K \mathbf{n}$, and $\{\{\mathbf{r}\}\} = \mathbf{r}|_K$ on e with $\mathbf{n}$ being the unit outward normal along Γ. C_{11} and $\mathbf{C}_{12}$ in (2.5)-(2.6) are edgewise defined, and their values may affect accuracy and stability of LDG methods as well as the matrix structures in the final algebraic system, see [2] and [6]. Detailed assumptions are made in Section 3 for C_{11} and $\mathbf{C}_{12}$.

With the definition of $(\hat{u}_h, \hat{\mathbf{q}}_h)$ in (2.5)-(2.6), by summing up all $K \in \mathcal{T}_h$, the LDG method becomes: finding $(u_h, \mathbf{q}_h) \in (V_h, \mathbf{M}_h)$, such that

$$a(\mathbf{q}_h, \mathbf{r}) + b(u_h, \mathbf{r}) = F(\mathbf{r}), \quad \forall\, \mathbf{r} \in \mathbf{M}_h, \tag{2.9}$$

$$-b(v, \mathbf{q}_h) + c(u_h, v) = G(v), \quad \forall\, v \in V_h. \tag{2.10}$$

Here

$$a(\mathbf{q}, \mathbf{r}) = \int_\Omega \mathbf{q} \cdot \mathbf{r}\, dx, \qquad c(u, v) = \int_{\mathcal{E}_i \cup \mathcal{E}_D} C_{11}[\![u]\!] \cdot [\![v]\!] ds,$$

$$\begin{aligned} b(u, \mathbf{r}) &= \sum_{K \in \mathcal{T}_h} \int_K u \nabla \cdot \mathbf{r}\, dx - \int_{\mathcal{E}_i} (\{\{u\}\} + \mathbf{C}_{12} \cdot [\![u]\!])[\![\mathbf{r}]\!] ds - \int_{\mathcal{E}_N} u \mathbf{r} \cdot \mathbf{n} ds, \\ &= -\sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \mathbf{r}\, dx + \int_{\mathcal{E}_i} (\{\{\mathbf{r}\}\} - \mathbf{C}_{12}[\![\mathbf{r}]\!]) \cdot [\![u]\!] ds + \int_{\mathcal{E}_D} u \mathbf{r} \cdot \mathbf{n} ds, \end{aligned}$$

$$F(\mathbf{r}) = \int_{\mathcal{E}_D} g_D \mathbf{r} \cdot \mathbf{n} ds, \qquad G(v) = \int_{\mathcal{E}_D} C_{11} g_D v ds + \int_{\mathcal{E}_N} v \mathbf{g}_{\mathbf{N}} \cdot \mathbf{n} ds.$$

The scheme (2.9)-(2.10) can also be written in a more compact form: finding $(u_h, \mathbf{q}_h) \in (V_h, \mathbf{M}_h)$, such that

$$\mathcal{A}(\mathbf{q}_h, u_h; \mathbf{r}, v) = \mathcal{F}(\mathbf{r}, v) \qquad \forall\, (v, \mathbf{r}) \in (V_h, \mathbf{M}_h) \tag{2.11}$$

with $\mathcal{A}(\mathbf{q}, u; \mathbf{r}, v) = a(\mathbf{q}, \mathbf{r}) + b(u, \mathbf{r}) - b(v, \mathbf{q}) + c(u, v)$ and $\mathcal{F}(\mathbf{r}, v) = F(\mathbf{r}) + G(v)$.

The only difference between the standard ([2]) and the local-structure-preserving ([6]) LDG methods is in the choice of the discrete spaces $(V_h, \mathbf{M}_h)$. For the standard LDG method, $(V_h, \mathbf{M}_h) = (V_h^{k,STD}, \mathbf{M}_h^{k,STD})$ is used, with

$$\begin{aligned} V_h^{k,STD} &= \{u \in L^2(\Omega) : u|_K \in P^k(K), \forall K \in \mathcal{T}_h\}, \\ \mathbf{M}_h^{k,STD} &= \{\mathbf{q} \in [L^2(\Omega)]^d : \mathbf{q}|_K \in [P^k(K)]^d, \forall K \in \mathcal{T}_h\}, \end{aligned}$$

where $P^k(K)$ is the set of polynomials of degree at most k on K. For the LSP-LDG method, we take $(V_h, \mathbf{M}_h) = (V_h^{k,LSP}, \mathbf{M}_h^{k,LSP})$, with

$$\begin{aligned} V_h^{k,LSP} &= \{u \in L^2(\Omega) : u|_K \in P^k(K), \triangle u|_K = 0, \forall K \in \mathcal{T}_h\}, \\ \mathbf{M}_h^{k,LSP} &= \{\mathbf{q} \in [L^2(\Omega)]^d : \mathbf{q}|_K \in [P^k(K)]^d, \nabla \cdot \mathbf{q}|_K = 0, \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Note that $\mathbf{q}_h$ can be solved locally in terms of u_h, the size of the final algebraic system of the LDG method therefore depends only on the dimension of V_h. In [6], another discrete space pair, $(V_h, \mathbf{M}_h) = (V_h^{k,LSP}, \widetilde{\mathbf{M}}_h^{k,LSP})$ with $\widetilde{\mathbf{M}}_h^{k,LSP} = \mathbf{M}_h^{k,STD}$, is also considered in the LSP-LDG method. In both cases, the approximating functions in V_h for LSP-LDG methods are piecewise harmonic polynomials, and such functions satisfy the differential equations exactly in each mesh element K.

The following is a summary of the results for the standard and the local-structure-preserving LDG methods related to the contents of this note ([2, 6]).

- (Solvability) The LDG method with any of the above discrete space pairs is uniquely solvable when $C_{11} > 0$ on $\mathcal{E}_i \cup \mathcal{E}_D$. The positiveness of C_{11} can be furthered relaxed as in [4] for the standard LDG method.

- (Error estimates) Assume the meshes $\{\mathcal{T}_h\}_h$ are regular (see Section 3). For the sufficiently smooth exact solution $(u, \mathbf{q})$ of (2.2), with $\mathbf{C}_{12} = O(1)$,

  - and with $C_{11} = O(1)$ or $O(h^{-1})$, the LDG method equipped with any of the above discrete space pairs (with the index k) satisfies the following error estimate

    $$|(\mathbf{q} - \mathbf{q}_h, u - u_h)|_{\mathcal{A}} = O(h^k).$$

Here

$$|(\mathbf{q},u)|_{\mathcal{A}} = \left(\|\mathbf{q}\|^2_{L^2(\Omega)} + \int_{\mathcal{E}_i\cup\mathcal{E}_D} C_{11}|[\![u]\!]|^2 ds\right)^{1/2} \tag{2.12}$$

and it defines an energy norm.

– the numerical solutions $u_h \in V_h^{k,STD}$ computed by the standard LDG method are proved to be optimal in the L^2 norm when $C_{11} = O(h^{-1})$ with $||u-u_h||_{L^2(\Omega)} = O(h^{k+1})$, and sub-optimal when $C_{11} = O(1)$ with $||u-u_h||_{L^2(\Omega)} = O(h^{k+1/2})$. Numerical results however show optimal convergence rates for both choices of C_{11}, see [2, 6]. Such optimal error estimates are also indicated by the numerical experiments for the LSP-LDG approximation $u_h \in V_h^{k,LSP}$ in the L^2 norm [6].

- (Computational complexity) Since the size of the final algebraic system of the LDG method only depends on the dimension of the space V_h, by incorporating the a priori knowledge of the exact solution to this discrete space, the LSP-LDG method results in a smaller linear system, especially when higher degree polynomials are used as approximations, and therefore has lower computational complexity. In particular, the dimension of the local-structure-preserving space $V_h^{k,LSP}$ on each element $K \in \mathcal{T}_h$ is $2k+1$ which depends on k linearly, whereas the dimension of the standard polynomial space $V_h^{k,STD}$ on K is $(k+2)(k+1)/2$, which depends on k quadratically. Indeed, this local-structure-preserving approximating space $V_h^{k,LSP}$ can be used in any of the DG methods discussed in [1] to provide high order numerical methods for the Laplace equation with low computational complexity. The actual cost efficiency of such methods certainly needs additional investigation.

3 Error estimates in the L^2 and the energy norms

In this section, the missing optimal L^2 error estimate from [6] will be established for the LSP-LDG method when $C_{11} = O(1/h)$. Though one can still work with the flux formulation of the bilinear form of the scheme (2.9)-(2.10), the proof of this section is instead based on its primal formulation ([1]), and this leads to a more concise analysis for the L^2 error estimate. In particular, in the primal formulation, the scheme is bounded and coercive with respect to a new energy norm $|\cdot|_{\mathcal{B}}$ (defined in (3.6)), and it is also consistent and adjoint consistent. Together with the approximation properties ([6]) of the locally harmonic polynomial functions for the solution u of the Laplace equation, the optimal error estimates in both the L^2 norm and the energy norm $|\cdot|_{\mathcal{B}}$ will follow. We will comment the L^2 error analysis based on the flux formulation at the end of this section, We first start with the notations and the assumptions needed in this section. For $\mathbf{C}_{12}$ and C_{11} in (2.5)-(2.6), it is assumed that $||\mathbf{C}_{12}||_{L^\infty(\mathcal{E}_i)} < \eta^\star < \infty$, $C_{11} = \eta_e h_e^{-1}$ on $e \in \mathcal{E}_i \cup \mathcal{E}_D$ with η_e as a constant, and $\min_{e\in\mathcal{E}_i\cup\mathcal{E}_D} \eta_e > \eta_0 > 0$, $\max_{e\in\mathcal{E}_i\cup\mathcal{E}_D} \eta_e < \eta_\infty < \infty$. And η_0, η_∞ and $\eta^\star$ are constants independent of h. Moreover, the meshes $\{\mathcal{T}_h\}_h$ are regular, that is, there exists a positive constant σ independent of h such that

$$\frac{h_K}{\rho_K} \le \sigma, \qquad \forall\ K \in \mathcal{T}_h \tag{3.1}$$

where ρ_K is the diameter of the largest circle B_K inscribed in K, and K is star-shaped with respect to B_K. If more general quadrilateral elements Ks are used, (3.1) needs to be slightly revised (p.247 in [3]). In order to get the optimal error estimate in the L^2 norm, the full elliptic regularity is assumed,

$$||\psi||_{2,\Omega} \le C_r ||f||_{0,\Omega}, \qquad \forall f\ \in L^2(\Omega) \tag{3.2}$$

with the constant C_r solely depending on Ω, for the adjoint problem of (2.1):

$$-\triangle\,\psi = f \quad \text{in } \Omega, \quad \psi|_{\Gamma_D} = 0, \quad \frac{\partial\psi}{\partial\mathbf{n}}|_{\Gamma_N} = 0. \tag{3.3}$$

Here $H^m(D)$ denotes the Sobolev spaces of order m on $D \subset \mathbb{R}^d$, with the usual norm $||\cdot||_{m,D}$ and $H^0(D) = L^2(D)$. We also use $V = \{v \in H^{k+1}(\Omega) : \triangle v = 0 \text{ in } \Omega\}$ and $V_h = V_h^{k,LSP}$. For any $v \in V_h$, $\nabla_h v$ represents its piecewise-defined gradient, and the collection of all such functions is $\nabla_h V_h$. In addition, two assumptions are made to $\mathbf{M}_h$, the discrete space for the auxiliary variable $\mathbf{q} = \nabla u$:

(i) Inclusion relation:

$$\nabla_h V_h \subset \mathbf{M}_h. \tag{3.4}$$

(ii) Inverse inequality: there exists a constant $C > 0$, depending on σ (and possibly the index k) and independent of $K \in \mathcal{T}_h$ and h, such that

$$||\mathbf{r}||_{0,\partial K} \le C h_K^{-1/2} ||\mathbf{r}||_{0,K}, \qquad \forall\ \mathbf{r} \in \mathbf{M}_h. \tag{3.5}$$

When $V_h = V_h^{k,LSP}$, examples of $\mathbf{M}_h$ satisfying (3.4) and (3.5) include $\mathbf{M}_h^{k,LSP}$, $\mathbf{M}_h^{k-1,LSP}$, $\widetilde{\mathbf{M}}_h^{k,LSP}$, and $\widetilde{\mathbf{M}}_h^{k-1,LSP}$.

Different from (2.12) which was used in [2] and [6], a new energy norm $|\cdot|_{\mathcal{B}}$

$$|v|_{\mathcal{B}}^2 = ||\nabla_h v||_{0,\Omega}^2 + |v|_*^2, \qquad \text{with} \qquad |v|_*^2 = \sum_{e\in\mathcal{E}_i\cup\mathcal{E}_D} h_e^{-1} ||[\![v]\!]||_{0,e}^2 \tag{3.6}$$

is introduced for $V(h) = V_h + V$. Here $||\cdot||_{0,e}$ is the L^2 norm of a function over an edge e. Note $|\cdot|_{\mathcal{B}}$ does not involve the space $\mathbf{M}_h$. The analysis in this section is closely related to the one in [1] which is to study the standard LDG method in its primal formulation. The main difference is in the definition of energy norms. With $|v|_{\mathcal{B}}$, we avoid more involved arguments which otherwise would be needed due to the use of the local-structure-preserving discrete spaces. Throughout the note, the letter C denotes a generic constant (whose value at different occurrences may vary) independent of mesh size h, but possibly dependent of the shape regularity parameter σ of the mesh, the index k (which is often the polynomial degree) for discrete spaces, the domain Ω, and η_0, η_∞ and $\eta_\star$ related to the numerical fluxes.

To carry out the analysis, we first establish several lemmas.

Lemma 3.1 (**Primal formulation**)**.** *The LSP-LDG method in its flux formulation* (2.9)-(2.10) *can be rewritten into its primal formulation: Looking for $u_h \in V_h$ such that*

$$\mathcal{B}(u_h, v) = \mathcal{S}(v; g_D, \mathbf{g}_N), \qquad \forall\, v \in V_h, \tag{3.7}$$

where

$$\mathcal{B}(w, v) = \int_\Omega (\nabla_h w + t([\![w]\!]))\,(\nabla_h v + t([\![v]\!]))\,dx + \int_{\mathcal{E}_i \cup \mathcal{E}_D} C_{11}[\![w]\!] \cdot [\![v]\!] ds, \tag{3.8}$$

$$\begin{aligned}\mathcal{S}(v; g_D, \mathbf{g}_N) &= -\int_\Omega \gamma(g_D)\,(\nabla_h v + t([\![v]\!]))\,dx + \int_{\mathcal{E}_D} C_{11} g_D v ds + \int_{\mathcal{E}_N} v \mathbf{g_N} \cdot \mathbf{n} ds, \\ &= -\int_\Omega \gamma(g_D)\,(\nabla_h v + t([\![v]\!]))\,dx + G(v).\end{aligned} \tag{3.9}$$

In addition

$$\mathbf{q}_h = \nabla_h u_h + t([\![u_h]\!]) + \gamma(g_D). \tag{3.10}$$

Here $t([\![v]\!]) = \alpha([\![v]\!]) + \beta(\mathbf{C}_{12} \cdot [\![v]\!])$*, and* $\alpha : [L^2(\mathcal{E}_i \cup \mathcal{E}_D)]^d \mapsto \mathbf{M}_h$*,* $\beta : L^2(\mathcal{E}_i) \mapsto \mathbf{M}_h$*,* $\gamma : L^2(\mathcal{E}_D) \mapsto \mathbf{M}_h$ *are lifting operators, defined as follows:* $\forall\, \mathbf{r} \in \mathbf{M}_h$*,*

$$\begin{aligned}\int_\Omega \alpha(\mathbf{z}) \cdot \mathbf{r} dx &= -\int_{\mathcal{E}_i \cup \mathcal{E}_D} \mathbf{z} \cdot \{\{\mathbf{r}\}\} ds, \\ \int_\Omega \beta(v) \cdot \mathbf{r} dx &= \int_{\mathcal{E}_i} v[\![\mathbf{r}]\!] ds, \\ \int_\Omega \gamma(v) \cdot \mathbf{r} dx &= \int_{\mathcal{E}_D} v \mathbf{r} \cdot \mathbf{n} ds.\end{aligned}$$

Proof. Start with (2.9), for any $\mathbf{r} \in \mathbf{M}_h$, one has

$$\begin{aligned}\int_\Omega \mathbf{q}_h \cdot \mathbf{r} ds &= \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \cdot \mathbf{r} dx - \int_{\mathcal{E}_i} (\{\{\mathbf{r}\}\} - \mathbf{C}_{12}[\![\mathbf{r}]\!]) \cdot [\![u_h]\!] ds - \int_{\mathcal{E}_D} u_h \mathbf{r} \cdot \mathbf{n} ds + \int_{\mathcal{E}_D} g_D \mathbf{r} \cdot \mathbf{n} ds, \\ &= \int_\Omega \nabla_h u_h \cdot \mathbf{r} dx - \int_{\mathcal{E}_i \cup \mathcal{E}_D} [\![u_h]\!] \cdot \{\{\mathbf{r}\}\} ds + \int_{\mathcal{E}_i} \mathbf{C}_{12} \cdot [\![u_h]\!][\![\mathbf{r}]\!] ds + \int_{\mathcal{E}_D} g_D \mathbf{r} \cdot \mathbf{n} ds, \\ &= \int_\Omega \big(\nabla_h u_h + \alpha([\![u_h]\!]) + \beta(\mathbf{C}_{12} \cdot [\![u_h]\!]) + \gamma(g_D)\big) \cdot \mathbf{r} ds.\end{aligned}$$

The last inequality results from the definitions of lifting operators. Using the assumption on the inclusion relation (3.4), one obtains the expression of $\mathbf{q}_h$ in terms of u_h by (3.10). With the similar argument, (2.10) turns to

$$\int_\Omega \left(\nabla_h v + \alpha([\![v]\!]) + \beta(\mathbf{C}_{12} \cdot [\![v]\!])\right) \cdot \mathbf{q}_h ds + \int_{\mathcal{E}_i \cup \mathcal{E}_D} C_{11} [\![u_h]\!] \cdot [\![v]\!] ds = G(v) \tag{3.11}$$

for all $v \in V_h$. Now plugging (3.10) into (3.11), one concludes (3.7). $\square$

Lemma 3.2 (**Estimates for lifting operators).** *For $v \in V(h)$, one has*

$$||\alpha([\![v]\!])||_{0,\Omega}^2 \le C|v|_*^2, \tag{3.12}$$

$$||\beta(\mathbf{C}_{12} \cdot [\![v]\!])||_{0,\Omega}^2 \le C||\mathbf{C}_{12}||_{L^\infty(\mathcal{E}_i)}|v|_*^2. \tag{3.13}$$

The constant C depends on σ and k, and it is independent of h.

Proof. First for any edge $e \in \mathcal{E}_i \cup \mathcal{E}_D$, one defines a local lifting operator $\alpha_e : [L^2(e)]^d \mapsto \mathbf{M}_h$,

$$\int_\Omega \alpha_e(\mathbf{z}) \cdot \mathbf{r} dx = -\int_e \mathbf{z} \cdot \{\{\mathbf{r}\}\} ds, \qquad \forall \mathbf{r} \in \mathbf{M}_h. \tag{3.14}$$

Note that $\alpha_e(\mathbf{z})$ vanishes outside the union of one or two elements containing e, and $\alpha([\![v]\!]) = \sum_{e\in\mathcal{E}_i\cup\mathcal{E}_D} \alpha_e([\![v]\!])$ for any $v \in V(h)$. By taking $\mathbf{r} = \alpha_e(\mathbf{z}) \in \mathbf{M}_h$ in (3.14), one has

$$\begin{aligned} ||\alpha_e(\mathbf{z})||_{0,\Omega}^2 &= -\int_e \mathbf{z} \cdot \{\{\alpha_e(\mathbf{z})\}\} ds \le ||\mathbf{z}||_{0,e} ||\{\{\alpha_e(\mathbf{z})\}\}||_{0,e}, \\ &\le C h_e^{-1/2} ||\mathbf{z}||_{0,e} ||\alpha_e(\mathbf{z})||_{0,\Omega}. \end{aligned}$$

The last step is by the inverse inequality assumption (3.5) and by $c_0 h_K \le h_e \le c_1 h_K$ for $e \subset \partial K$, with the constants c_0, c_1 depending on σ, and C depending on σ and k. Now we have $||\alpha_e(\mathbf{z})||_{0,\Omega} \le C h_e^{-1/2} ||\mathbf{z}||_{0,e}$. And

$$\begin{aligned} ||\alpha([\![v]\!])||_{0,\Omega}^2 &= ||\sum_{e\in\mathcal{E}_i\cup\mathcal{E}_D} \alpha_e([\![v]\!])||_{0,\Omega}^2 \le 3 \sum_{e\in\mathcal{E}_i\cup\mathcal{E}_D} ||\alpha_e([\![v]\!])||_{0,\Omega}^2, \\ &\le C \sum_{e\in\mathcal{E}_i\cup\mathcal{E}_D} h_e^{-1} ||[\![v]\!]||_{0,e}^2 = C|v|_*^2. \end{aligned}$$

Based on the relation of the lifting operators $\alpha(\cdot)$ and $\beta(\cdot)$ discussed in [1] (such relation does not depend on the actual definitions of V_h and $\mathbf{M}_h$), the estimate (3.13) then follows. $\square$

The results in Lemma 3.1 and Lemma 3.2 depend little on the actual definitions of V_h and $\mathbf{M}_h$. In fact, Lemma 3.1 only needs the inclusion relation (3.4), and Lemma 3.2 needs the inverse inequality assumption (3.5) on $\mathbf{M}_h$.

Lemma 3.3 (Boundedness and coercivity). *The LSP-LDG method in its primal formulation* (3.7)-(3.9) *is bounded on* $V(h) \times V(h)$ *and coercive on* $V_h \times V_h$ *under the energy norm* $|\cdot|_{\mathcal{B}}$ *for any* $\eta_0 = \min_{\mathcal{E}_i \cup \mathcal{E}_D} \eta_e > 0$. *That is,*

- (Boundedness) *There exists a constant* $M > 0$ *depending on* η_∞, $\eta^\star$, σ *and* k, *such that*

$$|\mathcal{B}(w, v)| \leq M||w||_{\mathcal{B}}||v||_{\mathcal{B}}, \qquad \forall\, w, v \in V(h). \tag{3.15}$$

- (Coercivity) *There exists a constant* $\theta > 0$ *depending on* η_0, $\eta^\star$, σ *and* k, *such that*

$$\mathcal{B}(v, v) \geq \theta ||v||^2_{\mathcal{B}}, \qquad \forall\, v \in V_h. \tag{3.16}$$

Proof. Based on Lemma 3.2, the definition of $\mathcal{B}(w, v)$, and the assumptions on C_{11} and $\mathbf{C}_{12}$, the boundedness (3.15) of $\mathcal{B}(w, v)$ comes straightforwardly.

To get the coercivity, one starts with any $v \in V_h$,

$$\begin{aligned}
\mathcal{B}(v, v) &= \int_\Omega |\nabla_h v + t([\![v]\!])|^2 dx + \sum_{e \in \mathcal{E}_i \cup \mathcal{E}_D} \eta_e h_e^{-1} ||[\![v]\!]||^2_{0,e} \\
&\geq \int_\Omega |\nabla_h v + t([\![v]\!])|^2 dx + \eta_0 |v|^2_* \\
&\geq (1 - \epsilon)||\nabla_h v||^2_{0,\Omega} + (1 - 1/\epsilon)||t([\![v]\!])||^2_{0,\Omega} + \eta_0 |v|^2_*, \qquad (\forall\, \epsilon \in (0, 1)) \\
&\geq (1 - \epsilon)||\nabla_h v||^2_{0,\Omega} + (1 - 1/\epsilon)C_0 |v|^2_* + \eta_0 |v|^2_*, \qquad \text{(By Lemma 3.2)} \\
&= (1 - \epsilon)||\nabla_h v||^2_{0,\Omega} + (\eta_0 + (1 - 1/\epsilon)C_0)|v|^2_*
\end{aligned}$$

The positive constant C_0 depends on $\eta^\star$, σ and k. Now since ϵ can be chosen as close to 1 as one wants, for any $\eta_0 > 0$, choose $\epsilon = \epsilon_0 \in (0, 1)$ such that $\eta_0 + (1 - 1/\epsilon_0)C_0 \geq \eta_0/2$, the coercivity result (3.16) will then follow with $\theta = \min(1 - \epsilon_0, \eta_0/2)$.

□

We are now ready to state the main error estimate results.

Theorem 3.4 (Error estimate in the energy norm). *Let* $u \in V$ *be the exact solution and* $u_h \in V_h = V_h^{k,LSP}$ *be the numerical solution. Then we have*

$$||u - u_h||_{\mathcal{B}} \leq C h^k ||u||_{k+1,\Omega}. \tag{3.17}$$

Here the constant $C > 0$ *depends on* η_0, η_∞, $\eta^\star$, σ *and* k, *and is independent of* h. *As a consequence, for* $\mathbf{q} = \nabla u$ *and* $\mathbf{q}_h$ *by* (3.10), *we have*

$$||\mathbf{q} - \mathbf{q}_h||_{0,\Omega} \leq C h^k ||u||_{k+1,\Omega}. \tag{3.18}$$

Proof. Following the derivation of the method, one can easily see that the scheme is consistent, therefore the error estimate in the energy norm can be obtained directly based on the boundedness, coercivity of $\mathcal{B}(w, v)$ (Lemma 3.3) and the approximation properties of V_h to V established in [6]. That is, $\forall\, v \in V_h$,

$$\begin{aligned}\theta||v-u_h||_{\mathcal{B}}^2 &\leq \mathcal{B}(v-u_h, v-u_h), \qquad \text{(By coercivity)}\\ &= \mathcal{B}(v-u, v-u_h), \qquad \text{(By consistency)}\\ &\leq M||u-v||_{\mathcal{B}}||v-u_h||_{\mathcal{B}}, \qquad \text{(By boundedness).}\end{aligned}$$

Then we have

$$||v-u_h||_{\mathcal{B}} \leq \frac{M}{\theta}||u-v||_{\mathcal{B}}, \qquad \forall\, v \in V_h.$$

Therefore

$$||u-u_h||_{\mathcal{B}} \leq ||u-v||_{\mathcal{B}} + ||v-u_h||_{\mathcal{B}} \leq (1+\frac{M}{\theta})||u-v||_{\mathcal{B}}$$

and

$$||u-u_h||_{\mathcal{B}} \leq (1+\frac{M}{\theta})\inf_{v\in V_h}||u-v||_{\mathcal{B}} \leq Ch^k||u||_{k+1,\Omega}$$

The last inequality is from $\inf_{v\in V_h}||u-v||_{\mathcal{B}} \leq C_0h^k||u||_{k+1,\Omega}$, an estimate based on Lemma 3.2, and Corollary 3.3 in [6]. The constant C_0 depends on σ and k.

To get the estimate (3.18) for $||\mathbf{q}-\mathbf{q}_h||_{0,\Omega}$, first notice that for the exact solution u, one has $\beta(\mathbf{C}_{12}\cdot[\![u]\!]) = 0$ and

$$\begin{aligned}\alpha([\![u]\!]) + \gamma(g_D) &= -\int_{\mathcal{E}_i\cup\mathcal{E}_D}[\![u]\!]\cdot\{\{\mathbf{r}\}\}ds + \int_{\mathcal{E}_D} g_D\mathbf{r}\cdot\mathbf{n}ds\\ &= -\int_{\mathcal{E}_D} u\mathbf{n}\cdot\mathbf{r}ds + \int_{\mathcal{E}_D} g_D\mathbf{r}\cdot\mathbf{n}ds = 0\end{aligned}$$

therefore $\mathbf{q} = \nabla u + \alpha([\![u]\!]) + \beta(\mathbf{C}_{12}\cdot[\![u]\!]) + \gamma(g_D)$. Together with (3.10), one gets

$$\mathbf{q}-\mathbf{q}_h = \nabla_h(u-u_h) + \alpha([\![u-u_h]\!]) + \beta(\mathbf{C}_{12}\cdot[\![u-u_h]\!]). \tag{3.19}$$

This leads to

$$\begin{aligned}||\mathbf{q}-\mathbf{q}_h||_{0,\Omega} &= ||\nabla_h(u-u_h) + \alpha([\![u-u_h]\!]) + \beta(\mathbf{C}_{12}\cdot[\![u-u_h]\!])||_{0,\Omega}\\ &\leq ||\nabla_h(u-u_h)||_{0,\Omega} + C_1|u-u_h|_*, \qquad \text{(By Lemma 3.2)}\\ &\leq (C_1+1)||u-u_h||_{\mathcal{B}} \leq Ch^k||u||_{k+1,\Omega}, \qquad \text{(By (3.17))}\end{aligned}$$

C_1 and C are positive constants depending on η_0, η_∞, $\eta^\star$, σ and k, and independent of h.

□

Theorem 3.5 (**Error estimate in the L^2 norm).** *Let $u \in V$ be the exact solution and $u_h \in V_h = V_h^{k,LSP}$ be the numerical solution. Then we have*

$$||u - u_h||_{0,\Omega} \leq Ch^{k+1}||u||_{k+1,\Omega}.$$

The constant $C > 0$ depends on η_0, η_∞, $\eta^\star$, σ, k and Ω, and is independent of h.

Proof. Consider the adjoint problem of (2.1).

$$-\triangle \psi = u - u_h \quad \text{in } \Omega, \quad \psi|_{\Gamma_D} = 0, \quad \frac{\partial \psi}{\partial \mathbf{n}}|_{\Gamma_N} = 0. \tag{3.20}$$

One can easily verify

$$\mathcal{B}(v, \psi) = (u - u_h, v) \qquad \forall\, v \in V(h), \tag{3.21}$$

indicating that the LSP-LDG method is adjoint consistent. Let ψ_I denote the L^2 orthogonal projection of ψ onto $V_h^{1,STD} = V_h^{1,LSP}$. By taking $v = u - u_h$ in (3.21), one has

$$\begin{aligned}
||u - u_h||_{0,\Omega}^2 &= \mathcal{B}(u - u_h, \psi) \\
&= \mathcal{B}(u - u_h, \psi - \psi_I), \qquad \text{(By consistency and } \psi_I \in V_h) \\
&\leq M||u - u_h||_{\mathcal{B}}||\psi - \psi_I||_{\mathcal{B}}, \qquad \text{(By boundedness)} \\
&\leq MC_0 h|\psi|_{2,\Omega}||u - u_h||_{\mathcal{B}} \leq MC_0C_r h||u - u_h||_{0,\Omega}||u - u_h||_{\mathcal{B}}.
\end{aligned}$$

The last two inequalities are based on the approximation property of ψ_I to ψ ([3]) and the full regularity assumption (3.2) for (3.20), with the constant C_0 depending on σ and C_r on Ω. Together with Theorem 3.4, one gets the error estimate in the L^2 norm

$$||u - u_h||_{0,\Omega} \leq MC_0C_r h||u - u_h||_{\mathcal{B}} \leq Ch^{k+1}||u||_{k+1,\Omega}.$$

□

Note the analysis in this section is based on the primal formulation of the scheme, it only relies on the approximation properties of V_h, not those of $\mathbf{M}_h$, therefore the error estimates in both the energy ($||\cdot||_{\mathcal{B}}$) and the L^2 norms from Theorems 3.4-3.5 are optimal with respect to the approximation properties of V_h. For the L^2 error estimate, the key observation is $V_h^{1,LSP} = V_h^{1,STD}$, that is, $V_h^{1,LSP}$ can be used to approximate the solutions of both the Laplace equation and its adjoint problem. As for $\mathbf{M}_h$, only the assumptions (3.4)-(3.5) are needed.

The optimal L^2 error estimate in Theorem 3.5 indeed can also be established based on the flux formulation of the scheme with duality argument ([2]), and the analysis will be more involved than the one presented above. To carry out such analysis, besides the standard components, one would also need

$$V_h^{1,LSP} = V_h^{1,STD}, \qquad \mathbf{M}_h^{0,LSP} = \mathbf{M}_h^{0,STD},$$

and

$$||\nabla_h(u - u_h)||_{0,\Omega} = ||\mathbf{q} - \mathbf{q}_h||_{0,\Omega} + Ch^k||u||_{k+1,\Omega}.$$

The latter is suggested by Theorem 3.5, and it can be derived from (3.19), Lemma 3.2, and Corollary 3.3 in [6].

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