Dynamic Phase Transition and Finite-size Effects in a Periodically Driven Spatially Extended Bistable System

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Metastability and hysteresis

- Ferromagnets
- Ferroelectrics
- Electrochemical adsorbate layers
- Autocatalytic chemical reactions
- Liquid crystals

- **Generic periodic response in spatially extended bistable systems**
Kinetic Ising Model \( T < T_c \)

\[ \mathcal{H} = -J \sum_{<i,j>} s_i s_j - H(t) \sum_{i=1}^{L^2} s_i \]

\( s_i = \pm 1 \quad J > 0 \)

- \( L \times L \) lattice with periodic boundary conditions
- Single-spin-flip Glauber dynamics
- **Periodic** square-wave field of amplitude \( H \)

Half-period: \( t_{1/2} \)

Magnetization: \( m(t) = (1/L^2) \sum_i s_i(t) \)

\( T < T_c \quad H \to -H \)

\( t=0: \quad m=1 \quad \text{escape from metastable well:} \quad t=\tau: \quad m=0 \)

Lifetime: \( \langle \tau \rangle = \langle \tau(T,H) \rangle \)

- Transition between symmetric and asymmetric hysteresis loops (limit cycles)
Metastable decay through homogeneous nucleation

\[ T < T_c \quad H = |H| \rightarrow -|H| \]

- Avrami’s law (for large systems):

\[ m(t) \approx 2 \varphi_{ms}(t) - 1 \]

\[ \varphi_{ms}(t) = e^{-\ln 2(t/\langle\tau\rangle)^3} \]
Hysteresis and dynamic response

\[ \mathcal{H} = -J \sum_{<i,j>} s_i s_j - H(t) \sum_i s_i \]

- **Periodic** square-wave field of amplitude \( H \)
- **Half-period** \( t_{1/2} \); \( \Theta = t_{1/2} / \langle \tau \rangle \)

\( \Theta \gg 1 \) symmetric limit cycle
\( \Theta \ll 1 \) asymmetric limit cycle
What happens when $\Theta \sim 1$?

(when $t_{1/2}$ and $\langle \tau \rangle$ become comparable)
Dynamic order parameter

\[ Q = \frac{1}{2t_{1/2}} \int m(t) dt \]

- Period-averaged magnetization
  - \( \Theta \gg \Theta_c \) : \( |Q| \approx 0 \) dynamically *disordered* phase
  - \( \Theta \ll \Theta_c \) : \( |Q| \approx 1 \) dynamically *ordered* phase
  - \( \Theta \approx \Theta_c \approx 1 \) (\( t_{1/2} \approx <\tau> \)) large fluctuations
Local (dynamic) order parameter

\[ \Theta \ll 1 \quad \Theta \approx 1 \quad \Theta \gg 1 \]

\[ Q_i = \frac{1}{2 t_{1/2}} \int s_i(t) \, dt \]
Dynamic Phase Transition (DPT)

\[ X_Q^L = L^2 (\langle Q^2 \rangle_L - \langle |Q| \rangle_L^2) \]

- Order-parameter fluctuations
- Analogous to susceptibility
- Diverges at \( \Theta_c \)
The fourth-order cumulant

\[ U_L = 1 - \frac{\langle Q^4 \rangle}{3\langle Q^2 \rangle^2} \]

\[ \Theta_c = 0.918 \pm 0.005 \]
\[ U_L = 0.611 \pm 0.003 \]
Finite-size scaling

\[ \theta = \left| \frac{\Theta - \Theta_c}{\Theta_c} \right| \]

“reduced” period

\[ \langle |Q| \rangle_L = L^{-\beta/\nu} F_{\pm} (\theta L^{1/\nu}) \]

\[ X_{L}^{Q} = L^{\gamma/\nu} G_{\pm} (\theta L^{1/\nu}) \]
Critical exponents

\[ \langle |Q| \rangle_L \propto L^{-\beta/\nu} \]
\[ X^O_L \propto L^{\gamma/\nu} \]
\[ | \Theta_c(L) - \Theta_c | \propto L^{-1/\nu} \]

(\text{at } \Theta = \Theta_c)

\[ \beta/\nu = 0.126 \pm 0.005 \]
\[ \gamma/\nu = 1.74 \pm 0.05 \]

(\nu = 0.95 \pm 0.15)
Full data collapse

- Supports the existence of scaling functions $F_{\pm}(x), G_{\pm}(x)$
Order-parameter histograms at $\Theta = \Theta_c$

time-series $\rightarrow P(Q)$
Scaling for the order-parameter distribution $P(|Q|)$

$$P_L(|Q|) = L^{\beta/\nu} \mathcal{P}(L^{\beta/\nu} | Q|)$$
Recent analytic result for DPT by Fujisaka et. al.

- Analyzed time-dependent Ginzburg-Landau equation
  \[ \frac{\partial m}{\partial t} = m - m^3 + h_o \sin(\pi \frac{t}{t_{1/2}}), \quad T < T_c \]

- Found bifurcation corresponding to the DTP and symmetry breaking limit cycle
  \[ \frac{\partial Q}{\partial t'} = -aQ - Q^3, \quad Q = \frac{1}{2t_{1/2}} \int_0^t m(t)\,dt \]

  \[ a > 0 : \text{ symmetric limit cycle } \quad Q_{st} = 0 \]
  \[ a < 0 : \text{ asymmetric limit cycle } \quad Q_{st} = \pm |Q_{sp}| \neq 0 \]
Dynamic Phase Diagram


\[ \Theta = \frac{t_{1/2}}{\langle \tau(T, H) \rangle} \]

(fixed \( t_{1/2} \))
1st order transition, tri-critical point?

Acharyya, PRE’99
Chakrabarti & Acharyya RMP’99

FIG. 1. Histograms of the normalized distributions of the dynamic order parameter $Q$ for different temperatures ($T = 0.20J/k_B$, $0.28J/k_B$, $0.30J/k_B$, and $0.40J/k_B$) and for the fixed value of the field amplitude $h_0 = 2.0J$. All the figures are plotted in the same scales.

FIG. 4. Temperature ($T$) variation of the fourth-order Binder cumulant. A deep minimum indicates that the transition is first order and the position of minimum is the transition point.
Metastable decay modes

\[ T < T_c \quad H \rightarrow -H \quad \text{(single field reversal)} \]

Multi-droplet (MD)

\[ L = 256 \]

\[ H = 2.0J \quad T = 0.35J \]

Single-droplet (SD)

\[ L = 32 \]

*Images show the decay process over time.*
Periodic response

- Symmetric limit cycle
- Dynamic phase transition (DPT)
- Asymmetric limit cycle

$t_{1/2} \gg \langle \tau(T,H) \rangle$
$t_{1/2} \sim \langle \tau(T,H) \rangle$
$t_{1/2} \ll \langle \tau(T,H) \rangle$

decreasing $T$

$L=180$

$L=16$

Stochastic resonance (SR)
“frozen”

$H=2.0J$
$t_{1/2}=50$ MCSS
Crossovers

I. MD $\rightarrow$ SD underlying metastable decay: dynamic spinodal (DSP)
Can be estimated analytically using standard nucleation theory: $T_{\text{DSP}}(H,L)$

II. SR $\rightarrow$ “dynamically frozen” regime (within SD):
Can be estimated analytically, assuming that SD switching is a Poisson process: $T_{\times}(H,L)$
$H = 2.0J$

$t_{1/2} = 50$ MCSS
Finite-size effects

\[ H = 2.0J \]
\[ t_{1/2} = 50 \text{ MCSS} \]

\[ Q = \frac{1}{2t_{1/2}} \int m(t) \, dt \]

\[ U_L = 1 - \frac{\langle Q^4 \rangle}{3 \langle Q^2 \rangle^2} \]
Finite-size effects (cont.)

\[ H = 1.8J \]
\[ t_{1/2} = 20 \text{ MCSS} \]

\[ Q = \frac{1}{2t_{1/2}} \int m(t)dt \]

\[ U_L = 1 - \frac{\langle Q^4 \rangle}{3\langle Q^2 \rangle^2} \]
Finite-size effects (cont.)

\[ H = 2.0J \]
\[ t_{1/2} = 500 \text{ MCSS} \]
Q histograms in the SD/SR regime

\[ L = 32 \]

\[ \Theta = \frac{t_{1/2}}{\langle \tau(T, H) \rangle} \]

\[ Q = \frac{1}{2t_{1/2}} \int m(t) dt \]

\[ P(Q) = \frac{e^{-\Theta}}{2} \delta(Q + 1) + \frac{\Theta}{2} e^{-\Theta|Q|} + \frac{e^{-\Theta}}{2} \delta(Q - 1) \]
RTD in the SD/SR regime

$L = 32$

\[ \Theta = \frac{t_{1/2}}{\langle \tau(T, H) \rangle} \]

peak height \( \propto e^{-n\Theta} \)
Order parameter histograms

\( L = 180 \)  

Dynamic phase transition (DPT)

\[ H = 2.0J \]
\[ t_{1/2} = 50 \text{ MCSS} \]

Decreasing \( T \)

\( L = 16 \)  

Stochastic resonance (SR)
Summary

- **Dynamic phase transition** (DPT) and **stochastic resonance** (SR) are generic responses in periodically driven bistable systems.
- **SR** is an important feature of small systems, but does not survive in the thermodynamic limit.
- **No** theory or simulations support the existence of a (dynamic) 1\textsuperscript{st} order transition and a corresponding tricritical point.