1. (30 points) Consider the following nonlinear programming problem:

\[
\min_{x_1, x_2} f(x) = \left( \frac{-2x_1 + x_2 + 2}{x_1 + 3x_2 + 4} \right)
\]

\[-x_1 + x_2 \leq 4
\]

\[2x_1 + x_2 \leq 14
\]

\[s.t \quad x_2 \leq 6
\]

\[x_1, x_2 \geq 0
\]

This problem is called a fractional program. Note the objective is not convex or concave. The objective is quasi-concave. Since Problem (1.1) minimizes a quasi-concave function over a bounded polyhedral set, we know that there exists an extreme point that is globally optimal. We also know

\[
\nabla f(x) = \begin{bmatrix}
-7x_2 - 10 \\
-7x_1 - 2 \\
\end{bmatrix}
\]

\[
\frac{(x_1 + 3x_2 + 4)^2}{(x_1 + 3x_2 + 4)^2}
\]

a. What are the first order necessary KKT conditions for Problem (1.1)?
b. Show that the point \(x^* = [7, 0]^T\) satisfies these conditions.
c. Does \(x^*\) satisfy the SOSC or SONC? Is \(x^*\) the unique global min?

2. Consider the following nonlinear programming problem:

\[
\max_{x_1, x_2} x_1^2 + 4x_1x_2 + x_2^2
\]

\[-x_1 + x_2 \leq 8
\]

\[s.t \quad -x_1 + 2x_2 \leq 4
\]

\[x_1, x_2 \geq 0
\]

a. What are the first order necessary conditions (KKT) for Problem (1.3)?
b. Do either of these points, \(x^* = [00]^T, \bar{x} = [44]^T\), satisfy the KKT conditions?
c. Do $x^*$ and $\bar{x}$ satisfy the SONC or SOSC? What can you say about the optimality of $x^*$ and $\bar{x}$? You may assume that you found all KKT points.

3. Consider the following nonlinear programming problem:

$$\min_{x_1, x_2, x_3} -3x_1 + x_2 - x_3^2$$

$$s.t. \quad x_1 + x_2 + x_3 \leq 0$$

$$-x_1 + 2x_2 + x_3^2 = 0$$

a. What are the first order necessary conditions (KKT) for this problem?

b. Somebody gives you a hint that the dual multipliers are $\lambda = [5/3 \quad -4/3]'$. Find a KKT point $x^*$.

c. Does $x^*$ satisfy the SONC or SOSC? What can you conclude about $x^*$ based on the SONC and SOSC?