Nonlinear Programming
Takehome Exam
Handed out November 8, 2008
Due in class Friday November 14, 2008
5% late penalty if not handed in class. On November 15.
Don’t miss class this week, see part III below.

Part I
Consider the following problem

\[ \begin{align*}
\min_x & \quad f(x) = x_1^2 - x_2^2 \\
\text{s.t.} & \quad x_1^2 + 2x_2^2 = 4 \\
& \quad x \in \mathbb{R}^2
\end{align*} \]

1. Find all Karush Kuhn Tucker (KKT) points of the above problem, i.e. all points 
   \((\bar{x} \in \mathbb{R}^2, \bar{\lambda} \in \mathbb{R})\) that satisfy Definition 5.2.1. Be careful there are more than one
   KKT points for this problem.
2. Consider the set \(S = \{x \mid x_1^2 + 2x_2^2 = 4\}\). What is \(T_S(\bar{x})\), the tangent cone of \(S\) for
   any given \(\bar{x} \in S\)?
3. Use Proposition 4.7.1 to state a necessary condition for \(\bar{x}\) to be a local minimum of \(f\) over the subset \(S\).
4. For each KKT point \((\bar{x}, \bar{\lambda})\) that you found in part 1, determine if \(\bar{x}\) satisfies the
   necessary condition for a local minimum in Proposition 4.7.1.
5. Are the multipliers \(\bar{\lambda}\) found in part 1, all geometric multipliers? Why or why not?
6. Find the dual function \(q(\lambda)\) in closed form. What is the solution of the dual problem?
7. What is the value of \(x\) that corresponds to the global minimum of this problem? Prove it.

PART II

Consider the following primal problem where \(f : \mathbb{R}^n \to \mathbb{R}, g_i : \mathbb{R}^n \to \mathbb{R}\),
and \(f\) and \(g_i, i = 1, \ldots, m\) are all smooth functions.

\[ f^* = \inf_x f(x) \]
\[ \text{s.t.} \quad g_i(x) \leq 0, i = 1, \ldots, m \]

and dual problem

\[ q^* = \sup_{u \geq 0} q(u) \]

where

\[ q(u) = \inf_x \left( f(x) + \sum u_i(g_i(x)) \right). \]
1. Prove the following theorem, if \( q^* = \infty \) (unbounded above) then the primal problem is infeasible.
2. Let \( x^* \) be the local minimum of the above primal problem and let the gradients of the active set, \( \nabla g_i(x^*) i \in A(x^*) \) be linearly independent where
   \[ A(x^*) = \{i \mid g_i(x^*) = 0\} \]. Prove using the results in the text that there exists \( u^* \in \mathbb{R}^m \) such that
   \[ g_i(x^*) \leq 0, \quad i = 1, \ldots, m \]
   \[ \nabla f(x^*) + \sum_{i=1}^{m} u_i^* \nabla g_i(x^*) = 0 \]
   \[ u^* \geq 0 \]
   \[ u_i^* (g_i(x^*)) = 0, \quad i = 1, \ldots, m \]

PART III
Attend all NLP classes this week (11/11 and 11/14) (worth 5% of total score). No excuses. All or nothing.