Introduction

As I take the queuing system course, I become interested in the queuing systems in our life, for example, the cafeteria in the school. In this project, I want to model an interesting queuing system I observed in some cafeteria on campus, and study the behaviour and performance of the queuing system at the cafeteria.

When I go to the Jazz Café in Darrin Communications Center, I noticed that there are two types of customers. Some of the customers quickly pick up some prepared food or drinks and join the check-out line, while some of them order sandwiches or coffee that take some time to be ready for pick-up. The prepared food usually serves in a limited amount, so it happened to me that the prepared sandwich is sold out. In that case, I would need to decide to get a fresh made sandwich or leave and make purchase somewhere else.

So I model such queuing network as a sandwich shop. Each day during the peak hour, 11:30 a.m. to 1:30 p.m., the sandwich shop have \( K \) prepared sandwich available in the refrigerator, and enough raw materials to make customized sandwich for all the arriving customers. The customer who wants customized sandwich is called customer of type 1 and arrives following a Poisson Process with rate \( \lambda_1 \). And the customer who wants to pick up a prepared sandwich and check out soon is called customer of type 2 and arrives following a Poisson process with rate \( \lambda_2 \).

When a type 1 customer arrives, he goes to the queue at the food counter, and wait for his turn when a staff will make his customized sandwich. Once he gets his sandwich, he proceeds to the checkout queue to pay for the food. There is one staff at the food counter, and his service time has exponential distribution with rate \( \mu_1 \). When a type 2 customer arrive, he goes to the queue in front of the refrigerator to wait for his turn to pick up a prepared sandwich. The time each type 2 customer needs to choose and take the prepared sandwich is also exponentially
distributed, but with a rate $\mu_2$. After taking his food, the type 2 customer proceeds to the checkout queue and pay for the food. If a type 2 customer arrives at the refrigerator and found that the prepared sandwiches are out of stock, he would join the queue of the type 1 customer and purchase a customized sandwich with probability $p$. Otherwise, he would left the sandwich shop without making any purchase. There is one cashier at the checkout, and he serves both types of costumers using exponentially distributed time with rate $\mu$.

Let $N_1(t)$ be the number of customized sandwich the shop sold by time $t$, and $N_2(t)$ be the number of prepared sandwich the shop sold by time $t$. When $N_2(t) < K$, the problem can be modelled as an open Jackson network shown in Figure 1. We call it *Phase-1 Network*.

And when $N_2(t) = K$, the problem is equivalent to an open Jackson network shown in Figure 2. We call it *Phase-2 Network*. 

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Figure 1: Jackson network for sandwich shop when $N_2(t) < K$

Figure 2: Jackson network for sandwich shop when $N_2(t) = K$
Data collection

Since our problem comes from the Jazz Café in Darrin Communications Center, I collected data from there to estimate the parameters. I arrived at the Jazz Café at 9:40 a.m. to purchase a fresh-made breakfast sandwich. Since 9:40 a.m. to 10:20 a.m. is a small peak hour for Jazz Café, I decided to spend half an hour there to count the number of customers and the food or drinks they get.

Upon my arrival, I found 8 salads, 11 prepared sandwiches, 7 wraps and 26 different types of cup cakes in the counter and the refrigerator. From 9:40 a.m to 10:10 a.m., there are 43 customers checked out at the counter. 12 of them bought prepared food from the refrigerator or the counter, e.g. prepared sandwiches, wraps, cup cakes and salads. 21 ordered fresh-made breakfast sandwiches, e.g. bagel with egg, cheese and sausage, bagel with peanut butter, croissants with butter. 34 ordered different types of drinks, e.g. coffee, ice coffee, soft drinks. There are still customers in the line waited to be served when I left at 10:10 a.m.

Since there are 3 staff in the Jazz Café at that time, one of them makes sandwiches, another makes coffees, and the third one checks out, we can ignore the coffee order and use the data for the check-out and sandwich-maker to estimate the parameters for our model. And to simplify the problem, we will make assumptions so that the parameters are relative integral numbers.

So we assume the arrival rate of the type 1 customer is $\lambda_1 = 42$ per hour, and the arrival rate of type 2 customer is $\lambda_2 = 24$ per hour. The sandwich-maker is idle sometimes, so we assume she can make 60 breakfast sandwiches per hour. The cashier works faster than the sandwich-maker, so we assume she can serve 90 customers per hour. The type 2 customers usually pick up their food in a few seconds. So we assume $\mu_2 = 6$ per minute. The total number of prepared food available for pick-up here is 8 + 11 + 7 + 26 = 52, hence we assume $K = 52$. At that time, I did not observe the out-of-stock of prepared food, so I simply assume $p = 50\%$. That is, if the prepared sandwiches are sold out, the type 2 customer has a half-half chance to leave the sandwich shop without making any purchase.

Queuing network analysis

Recall that our queuing network model has two phases: if $N_2(t) < K$, the network is in Phase-1 as it is shown in Figure 1 and if $N_2(t) = K$, the network is in Phase-2, as it is shown in Figure 2. Notice that in each phase, the network is an open Jackson network, so we analyze the network performance separately for each phase. We call the queue where type 1 customers get food 'queue 1', the queue where type 2 customers get food 'queue 2', and the check-out queue 'queue 3'.
Performance for Phase-1 network

We first look at the case when $N_2(t) < K$ and the network is in Phase-1.

In this phase, let $\gamma_i^{(1)}$ denote the external arrival rate for each node. Then by the assumptions in data collection, we have $\gamma_1^{(1)} = 42/60 = 0.7$ per minute, $\gamma_2^{(1)} = 24/60 = 0.4$ per minute, and $\gamma_3^{(1)} = 0$. Recall that in this phase, the customers leaving node 1 or node 2 all proceed to node 3. Now let $\lambda_i^{(1)}$ be the total mean flow rate into node $i$. By the traffic equation [1], we have

$$\lambda_1^{(1)} = \frac{\gamma_1^{(1)}}{1 - \rho_1^{(1)}} = 0.7, \quad \lambda_2^{(1)} = \frac{\gamma_2^{(1)}}{1 - \rho_2^{(1)}} = 0.4, \quad \lambda_3^{(1)} = \lambda_1^{(1)} + \lambda_2^{(1)} = 1.1.$$  

By Jackson’s theorem, each node behaves as an independent $M/M/1$ queue and the steady state distribution has product form. Let $X = (X_1, X_2, X_3)$ denote the state of the Phase-1 network system, where $X_i = x_i$ is the number of customers at node $i$ in steady state, for $i = 1, 2, 3$. By the assumptions in the Data collection section, we have the service rates for each server are $\mu_1^{(1)} = 60/60 = 1$ per minute, $\mu_2^{(1)} = 6$ per minute, and $\mu_3^{(1)} = 90/60 = 1.5$ per minute. Hence the traffic intensities are $\rho_1^{(1)} = \frac{\lambda_1^{(1)}}{\mu_1^{(1)}} = 0.7$, $\rho_2^{(1)} = \frac{\lambda_2^{(1)}}{\mu_2^{(1)}} = \frac{1}{15}$, and $\rho_3^{(1)} = \frac{\lambda_3^{(1)}}{\mu_3^{(1)}} = \frac{11}{15}$. So we have the probability mass function for the steady state distribution is

$$p_X(x_1, x_2, x_3) = \left[ (1 - \rho_1^{(1)}) (\rho_1^{(1)})^{x_1} \right] \left[ (1 - \rho_2^{(1)}) (\rho_2^{(1)})^{x_2} \right] \left[ (1 - \rho_3^{(1)}) (\rho_3^{(1)})^{x_3} \right]$$

$$= \left[ (1 - 0.7) (0.7)^{x_1} \right] \left[ (1 - \frac{1}{15}) (\frac{1}{15})^{x_2} \right] \left[ (1 - \frac{11}{15}) (\frac{11}{15})^{x_3} \right]$$

$$= \frac{28}{25} \frac{7^{x_1} 11^{x_2}}{15^{x_1 + x_2 + 1}}.$$  

The mean number in each queue separately is

$$\mathbb{E}[X_1] = \frac{\rho_1^{(1)}}{1 - \rho_1^{(1)}} = \frac{7}{3},$$

$$\mathbb{E}[X_2] = \frac{\rho_2^{(1)}}{1 - \rho_2^{(1)}} = \frac{1}{14},$$

$$\mathbb{E}[X_3] = \frac{\rho_3^{(1)}}{1 - \rho_3^{(1)}} = \frac{11}{4}.$$  

By Little’s Law, the mean delay at each node is
\[ \begin{align*}
\mathbb{E}[W_{1}^{(1)}] &= \frac{1}{\lambda_1^{(1)}} \mathbb{E}[X_1] = \frac{10}{3}, \\
\mathbb{E}[W_{2}^{(1)}] &= \frac{1}{\lambda_2^{(1)}} \mathbb{E}[X_2] = \frac{5}{28}, \\
\mathbb{E}[W_{3}^{(1)}] &= \frac{1}{\lambda_3^{(1)}} \mathbb{E}[X_3] = \frac{5}{2}.
\end{align*} \] (4)

Now for type 1 customers, their routes in the system are queues \((1, 3)\). So the mean time for a type 1 customer to make a purchase in Phase-1 network is

\[ \mathbb{E}[W_{T_1}^{(1)}] = \mathbb{E}[W_{1}^{(1)}] + \mathbb{E}[W_{3}^{(1)}] = \frac{35}{6} \approx 5.83 \text{ minutes}. \] (5)

And for a type 2 customer, his routes in the system consists of queues \((2, 3)\). So the mean time for a type 2 customer to make a purchase in Phase-1 network is

\[ \mathbb{E}[W_{T_2}^{(1)}] = \mathbb{E}[W_{2}^{(1)}] + \mathbb{E}[W_{3}^{(1)}] = \frac{75}{28} \approx 2.69 \text{ minutes}. \] (6)

The result is interesting because type 2 customer chose to get prepared sandwich usually because they are in a rush and do not have much time to wait for the customized sandwich to be made. However, since the bottle neck in the system is the check-out, and the type 2 customers do not have any priority at the check-out, they in fact did not save much time on average by getting a prepared sandwich. Because in this case, they spend lots of time in the line where they are mixed with type 1 customers.

This phenomena is common in our daily life when we are running late to class and want to pick up some food quickly when pass-by a cafeteria. In that case we usually would arrive the cafeteria a few minutes before the beginning of the class, which is in fact a small peak for the cafeteria. So we arrive and see a long line in front of the check-out. Then we need to decide weather to stay hungry or be late for the class. Because our intuition also provide the similar estimate as we did here. That is even if we can pick up the food in a few seconds, it still takes a long time to make the purchase because of the long waiting line at the check-out. So some of us would choose to leave the cafeteria without making a purchase.

So a piece of advice for the students is that be a few minutes earlier. And a piece of advice for the cafeteria to keep more customers is to add more checkout, or to provide a fast or prior check-out for type 2 customer.

**Performance for Phase-2 network**

When \( N_2(t) = K \), the network is in Phase-2 as it is shown in Figure 2 above.
In this phase, let \( \gamma_i^{(2)} \) denote the external arrival rate for each node. Then by the assumptions in data collection, we have \( \gamma_1^{(2)} = 42/60 = 0.7 \) per minute, \( \gamma_2^{(2)} = 24/60 = 0.4 \) per minute, and \( \gamma_3^{(2)} = 0 \). In this phase, the customers leaving node 1 still proceed to node 3, but the customers leaving node 2 have probability \( p \) to proceed to node 1 and get a customized sandwich, and have probability \( 1 - p \) to leave the system without making any purchase. So assuming \( p = 0.5 \), and \( \lambda_i^{(2)} \) be the total mean flow rate into node i. By the traffic equation \[1\], we have

\[
\lambda_2^{(2)} = \gamma_2^{(2)} = 0.4, \quad \lambda_1^{(2)} = \gamma_1^{(2)} + p \times \lambda_1^{(2)} = 0.9, \quad \lambda_3^{(2)} = \lambda_1^{(2)} = 0.9. \tag{7}
\]

By Jackson’s theorem, each node behaves as an independent \( M/M/1 \) queue and the steady state distribution has product form. Let \( Y = (Y_1, Y_2, Y_3) \) denote the state of the Phase-2 network system, where \( Y_i = y_i \) is the number of customers at node i in steady state, for \( i = 1, 2, 3 \). By the assumptions in the Data collection section, we have the service rates for each server are \( \mu_1^{(2)} = 60/60 = 1 \) per minute, \( \mu_2^{(2)} = 6 \) per minute, and \( \mu_3^{(2)} = 90/60 = 1.5 \) per minute. Hence the traffic intensities are \( \rho_1^{(2)} = \frac{\lambda_1^{(2)}}{\mu_1^{(2)}} = 0.9, \rho_2^{(2)} = \frac{\lambda_2^{(2)}}{\mu_2^{(2)}} = \frac{1}{15}, \) and \( \rho_3^{(2)} = \frac{\lambda_3^{(2)}}{\mu_3^{(2)}} = \frac{3}{5} \). So we have the probability mass function for the steady state distribution is

\[
p_Y(y_1, y_2, y_3) = \frac{1}{2^{y_1} - 13^{y_2} - 15^{y_3} + 15^{y_1} + y_2 + y_3 + 3}. \tag{8}
\]

We can also get the mean number in each queue separately as follows,

\[
\mathbb{E}[Y_1] = \frac{\rho_1^{(2)}}{1 - \rho_1^{(2)}} = 9, \\
\mathbb{E}[Y_2] = \frac{\rho_2^{(2)}}{1 - \rho_2^{(2)}} = \frac{1}{14}, \\
\mathbb{E}[Y_3] = \frac{\rho_3^{(2)}}{1 - \rho_3^{(2)}} = \frac{3}{2}. \tag{9}
\]

Comparing to the results for Phase-1 network, \( \mathbb{E}[Y_1] = 9 \) is much bigger than \( \mathbb{E}[X_1] = \frac{7}{3} \).
Especially if we consider the total mean flow rate into node 1 only increases from $\lambda_1^{(1)} = 0.7$ to $\lambda_1^{(2)} = 0.9$. This is because the traffic intensity increases from $\rho_1^{(1)} = 0.7$ to $\rho_1^{(2)} = 0.9$ which is very close to 1. On the other hand, the average number of customers at the check-out reduces from $E[X_3] = \frac{11}{4} = 2.75$ to $E[Y_3] = \frac{3}{2} = 1.5$. This make sense because some of the type 2 customer leave the system without making a purchase, they to not need to check out, hence there are less people go to the check-out counter.

Considering the total number of customers in system, we have for Phase-1 network, the mean number in system at steady state is

$$E[X] = E[X_1] + E[X_2] + E[X_3] = \frac{13}{84}.\quad (10)$$

And the mean number of customers in system for Phase-2 network in steady state is


Comparing (10) and (11), we can see that the number of customers in the system increases significantly. Which means that a more severe congestion happens in the Phase-2 network.

Now let us look at the mean delay for different customers. by Little’s Law, the mean delay at each node is

$$E[W_1^{(2)}] = \frac{1}{\lambda_1^{(2)}} E[Y_1] = 10,$$

$$E[W_2^{(2)}] = \frac{1}{\lambda_2^{(2)}} E[Y_2] = \frac{5}{28},$$

$$E[W_3^{(2)}] = \frac{1}{\lambda_3^{(2)}} E[Y_3] = \frac{5}{3}.\quad (12)$$

So for type 1 customers, their route is still queues (1,3). We then have their mean delay in system is

$$E[W_{T1}^{(2)}] = E[W_1^{(2)}] + E[W_3^{(2)}] = 11\frac{2}{3} \approx 11.67 \text{ minutes}.\quad (13)$$

And for type 2 customer who changed mind and decide to get a customized sandwich, their route consists of queues (2,1,3). So the average time they stay in the system is

$$E[W_{T2,1}^{(2)}] = E[W_1^{(2)}] + E[W_2^{(2)}] + E[W_3^{(2)}] = 11\frac{71}{84} \approx 11.85 \text{ minutes}.\quad (14)$$

For the type to customer who decide not to purchase anything at the sandwich shop, their route only consists of queue 2 itself. So the average time they stay in the system is
$$E[W^{(2)}_{T2,2}] = E[W^{(2)}_2] = \frac{5}{28} \approx 0.19 \text{ minutes.} \quad (15)$$

Comparing to the case for Phase-1 network, the time a customer need to make a purchase increases significantly. This matches our intuition. Since the bottle neck is now at the sandwich-maker, a piece of advice for the sandwich shop is that, if it runs out of the prepared sandwich, the service level could be improved by adding a sandwich-maker.

And for the student, if they arrive at the sandwich shop in a rush and want to get some food quickly, it might not be a good idea to switch choice when the prepared sandwiches are all sold out. Because it might take much longer than he expected.

**Phase transition of the network**

Now we can see the network has very different behaviours and performances in the two different phases. For example, in Phase-1 network, the bottle neck is at the check-out. While in Phase-2 network, the bottle neck is at the sandwich-maker. And in Phase-2 network, the congestion problem is more severe. In fact, if there is a long queue in the shop, some customer might chose to make purchase somewhere else, especially if they are in a rush. And the differences has impact on both the sandwich shop and the customers.

So it is very important to figure out when the network transforms from Phase-1 to Phase-2. We discuss this question in this section.

Let $T$ be the random variable representing the time of the phase transition in minutes. We want to find a probability distribution of $T$.

Recall that we use $N_2(t)$ to denote the number of prepared sandwiches that have been sold. And the phase transition happens when $N_2(t)$ first reaches $K$. On the other hand, If $C_2(t)$ denote the total number of type 2 customer arrived by time $t$, we have that $N_2(t) = C_2(t)$ when $C_2(t) < K$, and both of them reaches $K$ at the same time. So we can also observe the phase transition when $C_2(t)$ reaches $K$.

Now we can investigate the number of type 2 customers that has arrived by time $t$. Since the type 2 customers arrive follows an Poisson process with parameter $\gamma_2^{(1)} = 0.4$ per minute, the number of customers arrived between time 0 and time $t$ follows a Poisson distribution. In particular,

$$\Pr[C(t) - C(0) = k] = \frac{e^{-\gamma_2^{(1)} t} (\gamma_2^{(1)} t)^k}{k!} = \frac{e^{0.4t} (0.4t)^k}{k!}, \quad (16)$$

for all $k = 0, 1, 2, \ldots$. So by our assumption, $C(0) = 0$, we have $\Pr[C(t) = k] = \frac{e^{0.4t} (0.4t)^k}{k!}$. 

8
Now for any time $t > 0$, the event \{phase-transition happens before time $t$\} is equivalent to the event \{$C(t) \geq K$\}. That is, the event \{$T \leq t$\} is equivalent to the event \{$C(t) \geq K$\}. We then have

$$\Pr[T \leq t] = \Pr[C(t) \geq K] = \sum_{k=K}^{\infty} \Pr[C(t) = k] = \sum_{k=K}^{\infty} e^{0.4t} \frac{(0.4t)^k}{k!}.$$  \hfill (17)

Equation (17) gives the cumulative distribution function $F_T(t)$ for the phase-transition time $T$. In our case study, from the Data Collection section, we assumed that $K = 52$. So the cumulative distribution function for the phase-transition time $T$ is $F_T(t) = \sum_{k=52}^{\infty} e^{0.4t} \frac{(0.4t)^k}{k!}$.

We can numerically approximate the distribution function, and Figure 3 shows a plot of $F_T(t)$ while Table 1 shows the probabilities values that the 52 prepared sandwich would be sold out by time $t$.

**Conclusion and discussion**

In this project, we studied a queueing system which models a cafeteria on campus. The queueing systems has two phases. Each phase has different steady state performances. We
Table 1: $\Pr[T \leq t]$

<table>
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<th>$t$ (in minutes)</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
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<td>$\Pr[T \leq t]$</td>
<td>0.0000</td>
<td>$3.9 \times 10^{-9}$</td>
<td>$5.0 \times 10^{-7}$</td>
<td>$3.2 \times 10^{-5}$</td>
<td>0.0007</td>
<td>0.0071</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$t$ (in minutes)</th>
<th>100</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pr[T \leq t]$</td>
<td>0.0387</td>
<td>0.1303</td>
<td>0.3005</td>
<td>0.5184</td>
<td>0.7214</td>
<td>0.8649</td>
</tr>
</tbody>
</table>

compared the two different phases, and studied the distribution for the time when the phase would change.

By investigating the queuing system models of the cafeteria, we get better understanding for some phenomena in delay life, for example, the rush guy often could not save much time when running late. And knowing the mean time we might spent in the system also helps us make decisions, for example, 5 minutes earlier might avoid be in a long line and might make a big difference.

There are many ways we can improve our model. For example, in the data collection step, we only collected data on a particular day. If we make observations for several days, we might use those data to make a better estimation of the parameters. Another aspect we can analyse is the profit of the model. For example, give numerical support for our advice on how to improve the profit of the cafeteria.

References
