1. Consider the quadrature formula
\[ \int_0^h f(x) \, dx = \alpha_1 f(0) + \alpha_2 f(h/3) + \alpha_3 f(h) + E(h), \quad h = \text{constant} \]
(a) Find the interpolating polynomial \( \tilde{f}(x) \) of degree two (or less) that fits the data \((0, f(0)), (h/3, f(h/3))\) and \((h, f(h))\). Write the polynomial in terms of the Lagrange basis.
(b) Integrate the interpolating polynomial in part (a) to obtain the weights \((\alpha_1, \alpha_2, \alpha_3)\) in the numerical quadrature formula.

2. Consider the numerical quadrature formula
\[ \int_0^h f(x) \, dx = \alpha_1 f(0) + \alpha_2 f(2h/3) + E(h), \quad h = \text{constant} \]
(a) Find the weights \((\alpha_1, \alpha_2)\) so that \( E = 0 \) when \( f(x) = 1 \) and \( f(x) = x \), i.e. when \( f(x) \) is a polynomial of degree 1 or less.
(b) The error term has the form \( E = Kh^{p+2}f^{(p+1)}(c) \), where \( c \in [0, h] \). Consider monomials \( f(x) = x^2, x^3, \) etc., to find the positive integer \( p \) and the constant \( K \) in the error term.

3. Consider the integrals
\[ I_a = \int_{-1}^1 xe^{-2x} \, dx, \quad I_b = \int_{-2}^1 \frac{5x \, dx}{\sqrt{2 + 3x^2}}, \quad I_c = \int_{-1}^1 x \sin(x + 2) \, dx, \quad I_d = \int_1^3 x \ln(x^3) \, dx \]
Approximate \( I_a \) and \( I_b \) using 2-point Gaussian quadrature formulas, and approximate \( I_c \) and \( I_d \) using 3-point Gaussian quadrature formulas. (Use a change of variables for \( I_b \) and \( I_d \) following text exercise 4 on page 278.) Compare the approximate value of the integral with the exact value for each case.

4. The composite trapezoidal rule for \( m \) equally spaced subintervals is
\[ \int_a^b f(x) \, dx = \frac{h}{2} \left( f(a) + \sum_{j=1}^{m-1} f(x_j) + f(b) \right) - \frac{(b - a)h^2}{12} f''(c), \quad c \in [a, b] \]
where \( h = (b - a)/m \). Let \( f(x) = x/(x + 1) \), \( a = -1/2 \) and \( b = 2 \). Consider the error term in the composite trapezoidal rule above to determine the number of subintervals needed so that the absolute error in the numerical quadrature is less than \( 10^{-4} \).

5. Let \( I = \int_a^b f(x) \, dx \).
(a) Write a matlab function, \texttt{mySimpson} say, that outputs an approximation for \( I \) using the composite Simpson rule with \( m \) equally spaced subintervals. Your function should take input \( a, b, m \) and the function \( f \) (specified in an M-file).
(b) Let \( I = \int_{-1}^{1} \exp(-x) \sin(5x) \, dx \). Find approximations to \( I \) using your matlab function in part (a) with \( m = 10, 20 \) and \( 40 \). Compute the exact value for \( I \) (using Maple if you have to) and use it to find the absolute error in each approximation.