Sketching Conic Sections

Parabolas, ellipses & hyperbolas are called **conic sections** because they are all obtained by intersecting a plane with a double cone.

1. **Parabola**

   The standard equation for a vertically oriented parabola is
   
   $$x (y - y_0) = (x - x_0)^2$$

   $(x_0, y_0) =$ vertex; parabola widens as $|x|$ increases; $x > 0 \rightarrow$ opens up; $x < 0 \rightarrow$ opens down

   The standard equation for a horizontally oriented parabola is
   
   $$x (x - x_0) = (y - y_0)^2$$

   $(x_0, y_0) =$ vertex; parabola widens as $|x|$ increases; $x > 0 \rightarrow$ opens right; $x < 0 \rightarrow$ opens left

   **Note** completing the square may be needed to achieve these standard forms.

   **Example**

   $$y^2 + 2x + 4y + 2 = 0$$

   Complete square in $y$:
   
   $$y^2 + 4y + 4 - 4 + 2x + 2 = 0$$

   $$\Rightarrow (y + 2)^2 + 2x - 2 = 0 \Rightarrow (y + 2)^2 = -2(x - 1)$$

   **vertex** $(1, -2)$

   horizontally oriented
   opens left

   ![Graph of a parabola with vertex (1, -2) and opens to the left](image)
Ellipse

The standard equation of an ellipse is

\[
\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1
\]

\((x_0, y_0)\) center

\(a = x\)-dimension

\(b = y\)-dimension

Note completing the square may be needed to achieve this standard form

\[
9x^2 + 4y^2 - 18x + 8y - 23 = 0
\]

\[
9(x^2 - 2x) + 4(y^2 + 2y) - 23 = 0
\]

\[
9(x^2 - 2x + 1 - 1) + 4(y^2 + 2y + 1 - 1) - 23 = 0
\]

\[
9((x - 1)^2 - 1) + 4((y + 1)^2 - 1) - 23 = 0
\]

\[
9(x - 1)^2 + 4(y + 1)^2 = 36
\]

\[
\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1
\]

(1, -1) center

\(x\)-dimension = 2

\(y\)-dimension = 3
3) Hyperbola

The standard equation for a hyperbola is

\[ \pm \left( \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} \right) = 1 \]

\( (x_0, y_0) = \) intersection of asymptotes
+ selected : opens horizontally
- selected : opens vertically
\( a = \) horizontal displacement of vertices (if horiz)
\( b = \) vertical displacement of vertices (if vertical)
\[ \pm \frac{b}{a} = \text{asymptote slopes} \]

ex.

\[ \frac{(x-3)^2}{16} - \frac{(y-1)^2}{4} = 1 \]

opens horizontally
asym. intersect at \((3,1)\)
\( a = 4, b = 2 \)
asym slopes = \(\pm \frac{1}{2}\)
vertices 4 units left, right of \((3,1)\)

Note: as in the other two cases, completing the square may be necessary to achieve this standard form.