Calculus I Announcements

- Read sections 3.9-3.10
- Do all the homework for section 3.9 and problems 1,3,5,7 from section 3.10.
- The exam is in Thursday, October 22nd. The exam will cover sections 3.2-3.10, but we will discuss material beyond that before the exam.
Euler’s number $e$ is an important constant in Calculus. It is a transcendental number which means that it has no exact decimal representation, so we can only approximate it using decimals. It is approximately equal to 2.71828.

But in Calculus we need to use the exact constant so as to make the rules we develop precise. We do this by just using the letter $e$ to stand for the constant.

Associated with $e$ is the important function $y = e^x$, which is sometimes written as $y = exp(x)$. This function obeys all the usual rules of exponents. It also has the special property that $\frac{d}{dx} e^x = e^x$. 
Here are some equivalent definitions of the constant \( e \)

- \( e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \)
- \( e \) is the number such that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \)
- \( f(x) = \exp(x) \) is the unique function that satisfies the initial value problem:
  \[ f'(x) = f(x) \quad \text{and} \quad f(0) = 1 \]

Then \( e = \exp(1) \).

These definitions turn out to be equivalent, but we will not show why in this class.
General Derivative Rules for Exponents

Use \( \frac{d}{dx} e^x = e^x \) to show that

1. \( \frac{d}{dx} x^a = ax^{a-1} \) for values of \( a \neq 0 \). (Previously we had only shown this for positive integers.)

2. \( \frac{d}{dx} a^x = a^x \ln(a) \) for values of \( a > 0 \).
Logarithmic Differentiation

- We know \( \frac{d}{dx} x^n = nx^{n-1} \) where \( n \neq 0 \) is constant.
- We know \( \frac{d}{dx} a^x = a^x \ln(a) \) where \( a > 0 \) is constant.

**Question** So what is \( \frac{d}{dx} x^x \)?

**Answer** Use **Logarithmic Differentiation** whenever both the base and exponent both have variables. It works like this:

1. Write \( y = x^x = e^{\ln(x^x)} \)
2. Use the property of logs to simplify this to \( y = e^{x \ln(x)} \)
3. Differentiate both sides using the chain rule
4. There will be a term \( e^{x \ln(x)} \) in the resulting expression. Write this as \( x^x \)
Question  Find \( \frac{d}{dx} x^{\sin(x)} \)

A. \( \sin(x)x^{\sin(x)-1} \)

B. \( x^{\sin(x)} \cos(x) \)

C. \( x^{\sin(x)} \ln(x) \cos(x) \)

D. \( x^{\sin(x)} (\cos(x) \ln(x) + \sin(x)/x) \)

E. None of the above
Answer to Question  Find $\frac{d}{dx}x^{\sin(x)}$

A. $\sin(x)x^{\sin(x)-1}$

B. $x^{\sin(x)}\cos(x)$

C. $x^{\sin(x)}\ln(x)\cos(x)$

D. $x^{\sin(x)}(\cos(x)\ln(x) + \sin(x)/x)$ is the correct answer.

E. None of the above
Hyperbolic Trigonometric Functions

Define

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \]

\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \]

1. Show that \( \cosh^2(x) - \sinh^2(x) = 1 \)

2. Show that \( \frac{d}{dx} \cosh(x) = \sinh(x) \)

3. Show that \( \frac{d}{dx} \sinh(x) = \cosh(x) \)
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**Question** Find $\frac{d}{dx} \cosh(x) \sinh(x)$

A. $- \sinh(x) \cosh(x)$

B. $-1$

C. 1

D. $\cosh^2(x) + \sinh^2(x)$

E. None of the above
**Answer to Question**  Find $\frac{d}{dx} \cosh(x) \sinh(x)$

A. $- \sinh(x) \cosh(x)$  
B. $-1$  
C. $1$  

**D.** $\cosh^2(x) + \sinh^2(x)$ **is the correct answer.**  
E. None of the above  

**Note:** Recall formula is $\cosh^2(x) - \sinh^2(x) = 1$. 
Slide 9  **Inverse Hyperbolic Functions**

We can use the technique developed last time to find the derivatives of the inverse hyperbolic functions.

1. Use the relation \( \sinh(\sinh^{-1}(x)) = x \) to find a formula for \( \frac{d}{dx} \sinh^{-1}(x) \)

2. Use the relation \( \cosh(\cosh^{-1}(x)) = x \) to find a formula for \( \frac{d}{dx} \cosh^{-1}(x) \)
Idea of Related Rates Problems

Suppose $x$ and $y$ are functions of $t$. Given a relation between $x$ and $y$ we can differentiate the relation and the result will relate the rates $dx/dt$ and $dy/dt$

1. Suppose, $5x + 3y = 7$, find $dy/dt$ when $dx/dt = 4$

2. Suppose $y = x^3 - x$ and $dx/dt = 4$. Find $dy/dt$ when $x = 2$
Simple related rates questions already give you the equation as in the next question:

**Question** Suppose $x$ and $y$ are positive and depend on time $t$. If they are related by $x^2 + y^2 = 25$ and $dx/dt = 5$, find $dy/dt$ when $x = 3$.

A. $-60$

B. $-15/4$

C. $-12/5$

D. $380x$

E. None of the above
**Answer to Question** Suppose $x$ and $y$ are positive and depend on time $t$. If they are related by $x^2 + y^2 = 25$ and $dx/dt = 5$, find $dy/dt$ when $x = 3$.

A. $-60$

B. $-15/4$ is the correct answer.

C. $-12/5$

D. $380x$

E. None of the above
Related Rates: The balloon

Problem: Suppose a balloon is expanding so that its radius grows at a constant rate of 5 inches per minute. Find the rate of change of the volume when the radius is 2 inches.

Solution Technique:

• Label the variables and identify which depend on time.

• Find a formula to relate the variables.

• Differentiate the equation relating the variables to relate the rates.

• Substitute what is known and solve for what is desired.
Suppose that a drop of mist is a perfect sphere. Through condensation the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances, the radius of the raindrop increases at a constant rate.