Calculus I Announcements

- Read sections 3.8, 3.9
- Do the homework on the study guide for
  - All of section 3.8.
  - Section 3.9: 1, 3, 5, 7, 11, 17, 21, 25, 27, 29
- Exam 2 is in two weeks (October 22nd)
- The Next Lecture is on Tuesday, October 14th (Tuesday follows a Monday schedule)
Implicit Differentiation

Frequently it happens that we get an equation involving two variables, both $y$ and $x$ and we would like to find $\frac{dy}{dx}$ without being able to solve for $y$. The process that we use to do this is called implicit differentiation.

The following examples show how we differentiate an expression (one side of an equality) “implicitly” if $y$ is known to be a function of $x$.

- $\frac{d}{dx}(y + 5) = \frac{dy}{dx}$
- $\frac{d}{dx}(y^2 + x^2) = 2y\frac{dy}{dx} + 2x$
- $\frac{d}{dx}(y \sin(x)) = y \cos(x) + \sin(x)\frac{dy}{dx}$

Suppose $y$ is a function of $x$, and perform implicit differentiation on the following expressions.

1. $\frac{d}{dx}(xe^y + x^2)$
2. $\frac{d}{dx} x^3 y^2$
3. $\frac{d}{dx} \sin(y)x^2$
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**Question**  Suppose $y$ is a function of $x$ and find $\frac{d}{dx}yx$

A. $\frac{dy}{dx}$
B. $x\frac{dy}{dx} + xy$
C. $x\frac{dy}{dx} + y$
D. $x\frac{dy}{dx} + x$
E. None of the above
**Answer to Question** Suppose $y$ is a function of $x$ and find $\frac{dy}{dx}yx$

A. $\frac{dy}{dx}$

B. $x\frac{dy}{dx} + xy$

C. $x\frac{dy}{dx} + y$ is the correct answer.

D. $x\frac{dy}{dx} + x$

E. None of the above
Question  Suppose $y$ is a function of $x$ and find
\[
\frac{d}{dx} x \cos(y) + \ln(y)
\]

A. $\cos(y) + \frac{1}{y}$

B. $- \sin(y) \frac{dy}{dx} + \frac{1}{y} \frac{dy}{dx}$

C. $\cos(y) - x \sin(y) \frac{dy}{dx} + \frac{1}{y} \frac{dy}{dx}$

D. $\cos(y) \frac{dy}{dx} - x \sin(y) + \frac{1}{y} \frac{dy}{dx}$

E. None of the above
Answer to Question  Suppose $y$ is a function of $x$ and find $\frac{d}{dx}x \cos(y) + \ln(y)$

A. $\cos(y) + \frac{1}{y}$

B. $-\sin(y)\frac{dy}{dx} + \frac{1}{y \frac{dy}{dx}}$

C. $\cos(y) - x \sin(y)\frac{dy}{dx} + \frac{1}{y \frac{dy}{dx}}$ is the correct answer.

D. $\cos(y)\frac{dy}{dx} - x \sin(y) + \frac{1}{y \frac{dy}{dx}}$

E. None of the above
There are two ways we can express $y$ as a function of $x$.

- **Explicitly** In this case $y$ is written explicitly as a function of $x$ as in $y = x^3 + \sin(x)$.

- **Implicitly** In this case we have $y$ and $x$ related by an equation that we may not be able to easily solve for $y$.
  - For example $x^3 + y^3 = 6xy$ is a relation between $x$ and $y$ that gives $y$ as a function of $x$ implicitly.
  - But we can still think of $y$ as a function of $x$ on certain pieces
Implicit Functions Graphically

Make a graph of the folium of Descartes, $x^3 + y^3 = 6xy$.

1. Show that the graph passes through the point with coordinates $(3, 3)$.
2. Describe how the graph is related to the equation?
3. On which pieces is $y$ a function of $x$?
4. What does $dy/dx$ mean in terms of the graph?
The Implicit Differentiation Process

The implicit differentiation process is as follows.

- Suppose we know $y$ is a function of $x$ but we only have an implicit equation.
- Applying the operator $\frac{d}{dx}$ to both sides of the equality results in a new expression which contains $\frac{dy}{dx}$.
- We then solve the equation for $\frac{dy}{dx}$.

**Examples** Use implicit differentiation to find $\frac{dy}{dx}$ in terms of $x$ and $y$

1. $x^2 + y^2 = 25$
2. $x^3 + y^3 = 6xy$
Implicit differentiation can be used to find tangent lines and normal lines:

- The **tangent line** at a point on a graph is the line through the point and with slope equal to dy/dx.
- The **normal line** at a point on a graph is the line through the point and with slope perpendicular to dy/dx.
- Recall that if $m$ is the slope, the perpendicular slope is $-1/m$.

Try the following:

1. Find the tangent line to $x^3 + y^3 = 6xy$ at the point $(3, 3)$.
2. Find the normal line to $x^3 + y^3 = 6xy$ at the point $(3, 3)$. 

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**Question**  Find the tangent line to $x^2 + y^2 = 25$ at the point $(3, 4)$ by using implicit differentiation

A. $y = \frac{4}{3}(x - 3) + 4$

B. $y = \frac{3}{4}(x - 3) + 4$

C. $y = -\frac{3}{4}(x - 3) + 4$

D. $y = -\frac{4}{3}(x - 3) + 4$

E. None of the above
Answer to Question  Find the tangent line to $x^2 + y^2 = 25$ at the point $(3,4)$ by using implicit differentiation

A. $y = \frac{4}{3}(x - 3) + 4$

B. $y = \frac{3}{4}(x - 3) + 4$

C. $y = -\frac{3}{4}(x - 3) + 4$ **is the correct answer.**

D. $y = -\frac{4}{3}(x - 3) + 4$

E. None of the above
Implicit Differentiation

Question  Find the normal line to $x^2 + y^2 = 25$ at the point $(3, 4)$ by using implicit differentiation

A. $y = \frac{4}{3}(x - 3) + 4$

B. $y = \frac{3}{4}(x - 3) + 4$

C. $y = -\frac{3}{4}(x - 3) + 4$

D. $y = -\frac{4}{3}(x - 3) + 4$

E. None of the above
**Answer to Question**  Find the normal line to $x^2 + y^2 = 25$ at the point $(3, 4)$ by using implicit differentiation

A. $y = \frac{4}{3}(x - 3) + 4$  **is the correct answer.**

B. $y = \frac{3}{4}(x - 3) + 4$

C. $y = \frac{-3}{4}(x - 3) + 4$

D. $y = \frac{-4}{3}(x - 3) + 4$

E. None of the above
Finding $y''$ implicitly

**Example** Suppose $x^3 + y^3 = 54$ gives $y$ implicitly as a function of $x$. Find $y''$ by implicit differentiation.

The steps to do this are:

1. Find $y'$
2. Differentiate $y'$ to find $y''$
3. Substitute the result of item 1 into the result of item 2 and simplify.
4. In some cases, use the original equation to simplify further.
Derivatives of Inverses

• If we know the derivative of a function, we can find the derivative of its inverse by using the chain rule

• Show how to use this technique to derive the formula

\[
\frac{d}{dx} \ln(x) = \frac{1}{x}
\]

by using the knowledge that \(^1\)

\[e^{\ln(x)} = x\quad \text{and} \quad \frac{d}{dx} e^x = e^x\]

• Show how to use this technique to find

\[
\frac{d}{dx} \arcsin(x)
\]

by using the knowledge that \(^2\)

\[\sin(\arcsin(x)) = x\quad \text{and} \quad \frac{d}{dx} \sin(x) = \cos(x)\]

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\(^1\) for \(x > 0\)

\(^2\) for \(-1 \leq x \leq 1\)
Derivatives of Inverse Functions

Now, to get a general formula for the derivative an inverse,

We derive the formula for

\[
\frac{d}{dx} f^{-1}(x)
\]
in terms of \( f'(x) \).

- We know that \( f \left( f^{-1}(x) \right) = x \) for all \( x \) in the domain of \( f^{-1} \)
- Differentiating both sides and using the chain rule on the left we get

\[
f' \left( f^{-1}(x) \right) \frac{d}{dx} f^{-1}(x) = 1\]

- We want \( \frac{d}{dx} f^{-1}(x) \) so we solve for it to get

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f' \left( f^{-1}(x) \right)}
\]

Use this formula to derive the formula for the derivative of arctan(x) using the knowledge that \( \frac{d}{dx} \tan(x) = \sec^2(x) \).
Using techniques such as this, we can get derivatives of all the inverse trigonometric functions:

1. \( \frac{d}{dx} \arcsin(x) = \frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \)
2. \( \frac{d}{dx} \arccos(x) = \frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}} \)
3. \( \frac{d}{dx} \arctan(x) = \frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2} \)
4. \( \frac{d}{dx} \arccot(x) = \frac{d}{dx} \cot^{-1}(x) = \frac{-1}{1+x^2} \)
5. \( \frac{d}{dx} \arcsec(x) = \frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \)
6. \( \frac{d}{dx} \arccsc(x) = \frac{d}{dx} \csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}} \)

- Memorize formulas 1-4 for the exam.
Consider deriving the rule \( \frac{d}{dx} a^x = a^x \ln(a) \)
(where \( a > 0 \) is a constant).

- Use the definition of the derivative to show that
  \[
  \frac{d}{dx} a^x = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} = m(a)a^x
  \]
  where
  \[
  m(a) = \lim_{h \to 0} \frac{a^h - 1}{h}
  \]

- Define the number \( e \) to be the number so that
  \[
  m(e) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1
  \]
  which means that \( \frac{d}{dx} e^x = e^x \)

- Show that \( \frac{d}{dx} e^x = e^x \) is a special case of the general rule \( \frac{d}{dx} a^x = a^x \ln(a) \).

- Use implicit differentiation to derive the rule
  \[
  \frac{d}{dx} a^x = a^x \ln(a)
  \]
  from the knowledge that
  \[
  \frac{d}{dx} \ln(x) = \frac{1}{x}
  \]