Calculus I Announcements

- Find the course web page by searching on “RPI piper calculus”
- So far, you should have completed the problems from sections 1.1-1.6.
- Next, read sections 2.1, 2.2 and 2.3 and work the associated homework problems.
- Exam 1 is in 2 weeks
- You need to do ALL the homework problems from the course web page for the sections we cover in class.
The Limit of Secant Lines (2.1)

- The difference quotient is
  \[ \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{f(t) - f(a)}{t - a}, \text{ for } t \neq a \]

- The difference quotient is the formula to get the average rate of change.

- The difference quotient is also the formula to get the slope of the secant line.

- The derivative is the limit of the difference quotient, so it is the instantaneous rate of change.

- In symbols:
  \[ f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} = \lim_{t \to a} \text{ Slope of Secant Line} = \text{Slope of the Tangent Line} \]
The Limit of Secant Lines (2.1)

In pictures with $t$ approaching $a$ from the right this looks like:

$$f'(a) = \lim_{t \to a^+} \frac{f(t) - f(a)}{t - a} = \lim_{t \to a^+} \text{Slope of Right Secant Line}$$

= Slope of the Tangent Line
...and with \( t \) approaching \( a \) from the left this looks like:

\[
f'(a) = \lim_{t \to a^-} \frac{f(t) - f(a)}{t - a} = \lim_{t \to a^-} \text{Slope of Left Secant Line} = \text{Slope of the Tangent Line}
\]
The Tangent Line as a Limit (2.1)

- A secant line to the graph of $f$ is a line between two points on a graph.
- A tangent line to the graph of $f$ at a point is (intuitively) the line that just “kisses” the curve at the given point.
- The tangent line is determined as the limit of secant lines.
- The slope of the tangent line is determined as the limit of slopes of the secant lines...

  The Slope of the Tangent Line is: \[ \lim_{t \to a} \frac{f(t) - f(a)}{t - a} \]

- The slope of the tangent line is equal to the derivative of the function $f$ at that point...

  The Derivative of $f$ is: \[ f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} \]
Question  What are the average rates of change of 
$f(x) = 3x + 2$ over the intervals $[1, 1.1], [1, 1.01], [1, 1.001]$?

A. 0, 0, 0  
B. 3, .3, .03  
C. 5, .5, .05  
D. 3, 3, 3  
E. None of the above
Answer to Question  What are the average rates of change of $f(x) = 3x + 2$ over the intervals 
$[1, 1.1],[1, 1.01],[1, 1.001]$?

A. 0, 0, 0
B. 3, .3, .03
C. 5, .5, .05
D. 3, 3, 3  is the correct answer.
E. None of the above
Slide 7  iClicker

**Question**  What is the slope of the tangent line to $f(x) = 3x + 2$ at $x = 1$?

A. 3  
B. 5  
C. 2  
D. 1  
E. None of the above
Answer to Question  What is the slope of the tangent line to \( f(x) = 3x + 2 \) at \( x = 1 \)?

A. 3 is the correct answer.
B. 5
C. 2
D. 1
E. None of the above
Slide 8  **Finding Slopes of Other Functions**

Consider finding the slope of the tangent line to $f(x) = \sin(x)$ at $x = 0$?

- How would this task be formulated?
- What problems would be faced in carrying this out?
- How could these problems be overcome?

The resolution of these types of questions is what led to the formal definition of a derivative which is based on the concept of limit.

The key idea, is that we want to carefully consider what happens to the difference quotient as $x$ gets close 0, but $x \neq 0$. 

Chapter 2 is about taking limits.

We will use the notation:

$$\lim_{x \to c} f(x)$$

When reading this aloud, one says

*the limit as x goes to the number c of f of x*. 
Slide 10  Description of Limits (2.2)

- We write that the limit as \( x \) goes to \( c \) of \( f(x) \) is equal to \( L \) as

\[
\lim_{x \to c} f(x) = L
\]

- A good description for understanding what a limit means is: by taking \( x \) closer and closer to \( c \), but with \( x \neq c \), all the values of \( f(x) \) get closer and closer to \( L \), (or equal to \( L \)).

- Definition \( \lim_{x \to a} f(x) = L \) means that \( |f(x) - L| \) can be made arbitrarily small by taking \( x \) sufficiently close to \( c \), but not equal to, \( c \).

1. Explain how the description relates to the definition
2. Use the description and the definition to find \( \lim_{x \to 2} x^2 \)
3. Use the description and the definition to discuss

\[
\lim_{x \to 0} \sin \left( \frac{1}{x} \right)
\]

**Technical Note:** Whenever we write \( \lim_{x \to c} f(x) = L \) we assume \( f \) is defined in an open interval containing \( c \), except possibly at \( c \) itself.
Techniques for Finding Limits (2.2)

We used the definition of limits for investigating the limits on the previous page.
We now proceed to develop a set of general algebraic techniques that begin to formalize this process:

Here are some simple rules about limits that follow from the definition:

1. \( \lim_{x \to a} C = C \) where \( C \) is constant.
2. \( \lim_{x \to a} x = a \)
3. \( \lim_{x \to a} x^r = a^r \) where \( r \) is positive.
4. \( \lim_{x \to a} P(x) = P(a) \) if \( P \) is a polynomial
5. \( \lim_{x \to a} \sqrt{x} = \sqrt{a} \) for \( a \geq 0 \).
6. \( \lim_{x \to a} \ln(x) = \ln(a) \) for \( a > 0 \).
7. \( \lim_{x \to a} e^x = e^a \)
8. \( \lim_{x \to a} c^x = c^a \) for a constant \( c > 0 \).
9. \( \lim_{x \to a} \sin(x) = \sin(a) \) and similarly for cosine.
10. \( \lim_{x \to a} \tan(x) = \tan(a) \) as long as \( \tan(a) \) is defined (not infinity). Similarly for other trig functions.

Find the limits using the above rules:

\[
\begin{align*}
\lim_{x \to 3} x^3 + 4x & \quad \lim_{x \to \pi/6} \cos(x) & \quad \lim_{x \to 4} x^{1/5}
\end{align*}
\]
Limits: More Rules

If \( f \) and \( g \) are functions and \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) for real numbers \( L \) and \( M \) then the rules below may be proved using the definition and are used to find limits

1. \( \lim_{x \to a} (f(x) + g(x)) = L + M \)
2. \( \lim_{x \to a} (f(x) \cdot g(x)) = L \cdot M \)
3. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \) provided that \( M \neq 0 \).
4. \( \lim_{x \to a} (f(x))^r = L^r \) for any positive number \( r \)
5. \( \lim_{x \to a} \sin(f(x)) = \sin(L) \) and similarly for other trig functions with \( L \) is in their domain.
6. \( \lim_{x \to a} c^{f(x)} = c^L \) for \( c > 0 \)

Find the limits using the rules

\[
\lim_{x \to \pi/6} x^2 + 5x \sin(x) \quad \lim_{x \to 2} \frac{e^x + 3}{x + 4} \quad \lim_{x \to 5} \sqrt{\sin(x) + \sin(x^2) + 9\tan(x)}
\]
Limits from Graphs

- One can also figure out limits from simple graphs. The concept here, is given a graph, can we find the limit.

- On a graph, an “open circle” means that the function does **not** pass through the corresponding point.

- The idea is to look at what the function “should be” at the given point.

Draw a graph and try to find the limits from the graph.
One Sided Limits

Sometimes it is useful to find the limit from only one side.

- For the right hand limit only consider $x > c$
  
  Notationally: \( \lim_{x \to c^+} f(x) \)

- For the left hand limit only consider $x < c$
  
  Notationally: \( \lim_{x \to c^-} f(x) \)

Draw some graphical examples. Illustrate what left and right hand limits are if you are given a graph.

**Fact:** If

\[
\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L
\]

then

\[
\lim_{x \to c} f(x) = L
\]
Slide 15  **Piecewise Functions**

The rules for finding limits, do not apply to piecewise functions, so we have to find limits using the description. Given the function,

$$f(x) = \begin{cases} 
  x & \text{if } x < 2 \\
  5 & \text{if } x = 2 \\
  3 - x & \text{if } 2 < x < 3 
\end{cases}$$

determine if the following limits exist and if so, find their values

1. \( \lim_{x \to 2^-} f(x) \)
2. \( \lim_{x \to 2^+} f(x) \)
3. \( \lim_{x \to 2} f(x) \)
4. \( f(2) \)
Question For the function,
\[ f(x) = \begin{cases} 
\sin(x) & \text{if } x \leq 0 \\
x & \text{if } 0 < x < 1 \\
x^2 + 1, & \text{if } 1 \leq x
\end{cases} \]
which (if any) of the following is false.

A. \( \lim_{x \to 0^+} f(x) = 0 \)

B. \( \lim_{x \to 0} f(x) = 0 \)

C. \( \lim_{x \to 1^-} f(x) = 1 \)

D. \( \lim_{x \to 1} f(x) = 1 \)

E. All the above are true
Answer to Question  For the function,

\[ f(x) = \begin{cases} 
  \sin(x) & \text{if } x \leq 0 \\
  x & \text{if } 0 < x < 1 \\
  x^2 + 1, & \text{if } 1 \leq x 
\end{cases} \]

which (if any) of the following is false.

A. \( \lim_{x \to 0^+} f(x) = 0 \)

B. \( \lim_{x \to 0} f(x) = 0 \)

C. \( \lim_{x \to 1^-} f(x) = 1 \)

D. \( \lim_{x \to 1} f(x) = 1 \)  \textit{is the correct answer.}

E. All the above are true
Slide 17  **Infinite Limits**

- We say \( \lim_{x \to c} f(x) = \infty \) if the values of \( f(x) \) “increase” without bound as \( x \) gets close to \( c \) but \( x \) not equal to \( c \).
- We say \( \lim_{x \to c} f(x) = -\infty \) if the values of \( f(x) \) “decrease” without bound as \( x \) gets close to \( c \) but \( x \) not equal to \( c \).

Similar descriptions apply for left and right hand limits.

Use the above descriptions to find/discuss

1. \( \lim_{x \to 0} \frac{1}{x^2} \)
2. \( \lim_{x \to 3^+} \frac{1}{x - 3} \)
3. \( \lim_{x \to 3^-} \frac{1}{x - 3} \)
Slide 18  **Limits and Graphs**

Sketch the graph of a function $y = g(x)$ that satisfies the following:

- $g(1) = 0$
- $g(2) = 3$
- $g(3) = 4$
- $\lim_{x \to 1} g(x) = -3$
- $\lim_{x \to 2^-} g(x) = \infty$
- $\lim_{x \to 2^+} g(x) = -\infty$
- $\lim_{x \to 3^+} g(x) = -2$
- $\lim_{x \to 3^-} g(x) = 1$