HW2: Finite Element Analysis
Due 2pm on February 26; Assigned on Feb 19, 2016

1. **FE approximation for $u'$.** Consider a mesh on $[a, b]$:

   \[ a = x_0 < x_1 < \cdots < x_j < \cdots < x_N < x_{N+1} = b, \]

   with $I_j = [x_{j-1}, x_j]$ and $h_j = x_j - x_{j-1}$. Consider a finite dimensional space

   \[ W_h = \{ w : w \in C^0([a, b], w|_{I_j} \text{ is a linear polynomial}, \ w(a) = w(b) = 0 \}. \]

   For $u, v \in W_h$, one can convert a bilinear form $b(u, v) = (u', v)$ into an algebraic form $c^\top B d$. Here $c$ and $d \in \mathbb{R}^N$ are the coefficient vectors of $u$ and $v$, respectively, when they are expanded with respect to hat function basis of $W_h$; $B \in \mathbb{R}^{N \times N}$.

   1.1) Find the matrix $B$ explicitly.

   1.2) What is the value of $b(u, u)$ for any $u \in W_h$? Does your answer explain any of the properties of the matrix $B$?

2. **Weak derivative.** What is the $k$-th order weak derivative of $f(x) = \exp(|x|)$, with $k = 1, 2, \cdots$? Provide some reason /argument to support your answer.

3. **Variants of Friedrichs’ inequality.**

   3.1) Show

   \[ ||v||_{L^2(a,b)} \leq C \left( |\bar{v}| + ||v||_{H^1(a,b)} \right), \quad \forall v \in C^\infty(a, b). \]

   Here $\bar{v}$ is the average of $v$ over $[a, b]$, namely, $\bar{v} = \frac{1}{b-a} \int_a^b v(y)dy$. $C$ is a finite and positive constant, independent of $v \in C^\infty(a, b)$ and possibly depending on $a$ and $b$.

   3.2) Using the fact that $C^\infty(a, b) \cap L^2(a, b)$ is dense in $(L^2(a, b), ||\cdot||_{L^2(a,b)})$, and $C^\infty(a, b) \cap H^1(a, b)$ is dense in $(H^1(a, b), ||\cdot||_{H^1(a,b)})$, further show

   – The inequality (3) holds for all $v \in H^1(a, b)$. This gives one Friedrichs’ inequality.

   – With the same constant $C$, one has additional variants of Friedrichs’ inequalities,

   \[ ||v - \bar{v}||_{L^2(a,b)} \leq C ||v||_{H^1(a,b)}, \quad \forall v \in H^1(a,b), \]

   \[ ||v||_{L^2(a,b)} \leq C ||v||_{H^1(a,b)} \quad \forall v \in V = \{ v \in H^1(a,b), \bar{v} = 0 \}. \]

4. **A surprise!** Consider the first model problem discussed in class

   \[ -u''(x) = f(x), \quad x \in (0, 1), \quad u(0) = u(1) = 0, \]

   its weak formulation: look for $u \in W$, such that

   \[ (u', v') = (f, v), \quad \forall v \in W, \]

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and its finite element method with piecewise linear space: look for $U \in W_h$, such that

$$(U', v') = (f, v), \quad \forall v \in W_h. \quad (8)$$

The space $W$ is defined as

$$W = \{w : w \in C^0[0,1], w' \text{ is piecewise continuous and bounded, } w(0) = w(1) = 0\},$$

and $W_h$ is defined in (2) ($a = 0, b = 1$) with respect to the mesh in (1). Let $G_i \in W$ satisfy

$$(v', G'_i) = v(x_i), \quad \forall v \in W, \quad (9)$$

where $x_i$ is any given mesh node, $i = 1, 2, \cdots N$. Prove or verify that $G_i$ is given by

$$G_i(x) = \begin{cases} (1 - x_i)x & 0 \leq x \leq x_i \\ x_i(1 - x) & x_i \leq x \leq 1 \end{cases}. \quad (10)$$

You might notice that $G_i \in W_h$. Now by choosing $v = e = u - U$ in (9), show that

$$e(x_i) = (e', G'_i) = 0, \quad i = 1, \cdots N. \quad (11)$$

This implies $U$ is exactly equal to $u$ at the mesh node points. Additional notes: (1) $G_i$ is the Green's function for (6) associated with a delta function $\delta(x_i)$ at node $x_i$: $G_i$ satisfies $-G''_i = \delta(x_i)$ on $(0, 1)$, and $G_i(0) = G_i(1) = 0$. (2) This somewhat surprising fact is a true one-dimensional effect. (3) The technique of working with a Green's function is useful in proving pointwise error estimates in one or higher dimensions. (4) If your code for hw1 works, you may want to numerically confirm this new discovery about $U$. 