1. Consider the initial-value problem

\[ y' = f(x)g(y), \quad y(x_0) = y_0, \]  

where the functions \( f(x) \) and \( g(y) \) are continuous near \( x_0 \) and \( y_0 \), respectively. Show that

(i) If \( g(y_0) \neq 0 \), then there exists a unique solution of (1).

(ii) If \( g(y_0) = 0 \), and \( g(y) \neq 0 \) for all \( y \) close enough to \( y_0 \), then there exists a unique solution of (1) precisely when the integral

\[ \int_{y_0}^{y} \frac{d\eta}{g(\eta)} \]

diverges.

Find the solution in each case.

2. Consider the differential equation

\[ y' = \sqrt{|y|}. \]

Find all of its possible solutions that pass through the point \((x_0, 0)\), and discuss your findings.

3. Consider Riccati’s equation

\[ y' + g(x)y + h(x)y^2 = k(x). \]  

(i) If \( y = \phi(x) \) is any solution of (2), then show that the substitution \( y = \phi(x) + 1/z \) will lead to its general solution. Write this solution down explicitly in terms of quadratures (i.e., integrals).
(ii) Deduce from (i) that if \( \phi_1(x) \), \( \phi_2(x) \), and \( \phi_3(x) \) are three particular solutions of (2), then its general solution is given by

\[
\frac{y - \phi_2(x)}{y - \phi_1(x)} = C \frac{\phi_3(x) - \phi_2(x)}{\phi_3(x) - \phi_1(x)},
\]

where \( C \) is an arbitrary constant.

(iii) Show that the substitution

\[
y = \frac{u'}{h(x)u}
\]

will transform (2) into a linear second order equation.

4. Consider Clairaut’s equation

\[
y = xy' + g(y').
\]  

(i) Find a parametric solution \((x(p), y(p))\) of (3) in terms of the parameter \( p = y' \). Differentiating (3) once will help.

(ii) Find a family of straight-line solutions of (3).

(iii) Show that the solution found in (i) is the envelope of the family of straight lines found in (ii). (First show that the envelope of the family of curves given by an equation \( F(x, y, c) = 0 \) is obtained by eliminating \( c \) from this equation and the equation \( \partial_c F(x, y, c) = 0 \).)

5. Consider the system

\[
\dot{E} = P - \alpha E, \quad \dot{P} = E - \beta P, \quad \dot{D} = -\gamma D,
\]

where the overdot denotes differentiation with respect to the time-variable \( t \), and \( \alpha \), \( \beta \), and \( \gamma \) are non-negative parameters.

(i) Find the general solution of the system (4).

(ii) Find the \textit{stable} and \textit{unstable subspaces} of the origin, that is, the sets of all solutions of system (4) that approach the origin for \( t \to \infty \) and \( t \to -\infty \), respectively. What kind of geometric objects are these subspaces? How do they change with the parameters \( \alpha \), \( \beta \), and \( \gamma \)?
6. Let the function $k(x,t,z)$ be continuous, and let its partial derivative $\partial_z k(x,t,z)$ be continuous and uniformly bounded for $0 \leq t \leq x \leq a$ and all real $z$. Also, let the function $g(x)$ be continuous for $0 \leq x \leq a$. Show that the Volterra integral equation

$$u(x) = g(x) + \int_0^x k(x,t,u(t)) \, dt$$

possesses exactly one continuous solution in the interval $0 \leq x \leq a$.

7. Consider the equation $\dot{x} = Ax$, where $x$ is an $n$-dimensional real vector, and $A$ is a given $n \times n$ real matrix. Use Picard iterations to show that the general solution of this equation is $x = e^{At}x_0$, where $x_0$ is any constant vector, and

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

8. Investigate how far solutions of the initial-value problem

$$\dot{x} = x^2, \quad x(t_0) = x_0$$

can be extended.

9. In this exercise, you will show a theorem on extension of solutions by filling in the steps of the following outline:

(i) Define a function $f(x,y)$ to satisfy a local Lipschitz condition in the set $D \subset \mathbb{R}^2$ if for every point $(x_0,y_0) \in D$, there exists a neighborhood $U = U(x_0,y_0)$ and a number $L = L(x_0,y_0)$, such that the function $f$ satisfies the Lipschitz condition

$$|f(x,y) - f(x,\bar{y})| \leq L |y - \bar{y}|$$

(5)

on the intersection $D \cap U$.

Show that if the set $D$ is open and if $f \in C(D)$ has a continuous derivative $f_y$ in $D$, then $f$ satisfies a local Lipschitz condition in $D$.

(ii) Show that if $D$ is open and $f \in C(D)$ satisfies a local Lipschitz condition in $D$, then the initial-value problem

$$y' = f(x,y), \quad y(\xi) = \eta$$

(6)

locally has a unique solution for every $(\xi,\eta) \in D$. In other words, in some small enough neighborhood of every $\xi$, there exists a unique solution.
(iii) Let $f$ be defined in $D$ and let $\Phi = \{\phi_\alpha\}_{\alpha \in A}$ be a family of solutions for the initial-value problem (6) in the respective intervals $J_\alpha$, such that
$$\phi_\alpha(x) = \phi_\beta(x) \quad \text{for} \quad x \in J_\alpha \cap J_\beta \quad (\alpha, \beta \in A).$$

Show that there exists a unique solution $\phi$ of (6) in the interval $J = \bigcup_{\alpha \in A} J_\alpha$ with $\phi|_{J_\alpha} = \phi_\alpha$ for all $\alpha \in A$.

An immediate consequence is that if the initial-value problem (6) has at least one solution, and the uniqueness statement (7) holds for any two solutions, then there exists a solution of (6) that cannot be extended. All other solutions are restrictions of this solution.

REMARK: Note that $\xi \in J_\alpha$ for all $\alpha$.

HINT: First show $\phi$ is uniquely defined. Then show that $\phi$ is indeed a solution in $J$.

(iv) Let $D \subset \mathbb{R}^2$ and $f \in \mathcal{C}(D)$. If $\phi$ is a solution of the differential equation $y' = f(x, y)$ on the interval $\xi \leq x < b$, which is entirely contained in a compact set $A \subset D$, then show that $\phi$ can be extended as a solution to the closed interval $[\xi, b]$.

HINT: Show that boundedness of $f$ on $A$ implies the existence of the limit $\lim_{x \to b^-} \phi(x)$. Define $\phi(b)$ to be this limit, and show that the thus-obtained extended function $\phi$ is left-differentiable at $b$ and satisfies the differential equation.

(v) Let $D \subset \mathbb{R}^2$ and $f \in \mathcal{C}(D)$. Let $\phi$ be a solution of the differential equation $y' = f(x, y)$ on the interval $[\xi, b]$, let $\psi$ be a solution on the interval $[b, c]$, and let $\phi(b) = \psi(b)$. Show that
$$u(x) = \begin{cases} 
\phi(x) & \text{for} \quad \xi \leq x \leq b \\
\psi(x) & \text{for} \quad b < x \leq c 
\end{cases}$$

is a solution in the interval $[\xi, c]$.

HINT: Compare left-hand and right-hand derivatives.

(vi) Let $D \subset \mathbb{R}^2$ be open, and let $f \in \mathcal{C}(D)$ satisfy a local Lipschitz condition in $D$. Show that for every $(\xi, \eta) \in D$, the initial-value problem (6) has a solution $\phi$ which cannot be extended, and which both on the right and on the left of $\xi$ approaches the boundary of $D$ arbitrarily closely. Furthermore, show that the solution $\phi$ is uniquely determined, that is, all other solutions of (6) are restrictions of $\phi$.

REMARK: The statement that $\phi$ on the right of $\xi$ approaches the boundary of $D$ arbitrarily closely is defined as follows: If $G$ is the closure of the graph of $\phi$, and $G_+$ is the subset of the points $(x, y) \in G$ with $x \geq \xi$, then $G_+$ is not a compact subset of $D$.

An equivalent, but more intuitive, formulation is the following:
The solution \( \phi \) exists on the right of \( \xi \) in some interval \( \xi \leq x < b \) (with \( b = \infty \) permitted), and one of the following statements holds:

(a) \( b = \infty \); the solution exists for all \( x \geq \xi \).

(b) \( b < \infty \), \( \limsup_{x \to b^-} |\phi(x)| = \infty \); the solution “blows up.”

(c) \( b < \infty \), \( \limsup_{x \to b^-} \rho(x, \phi(x)) = 0 \), where \( \rho(x_0, y_0) \) is the distance of the point \( (x_0, y_0) \) from the boundary of \( D \); the solution “approaches the boundary arbitrarily closely.”

In fact, the above definition states that \( G_+ \) is either unbounded (case (a) or (b)), or else bounded and contains boundary points of \( D \).

HINT: For uniqueness, assume that the solution “splits” at some point \( x_0 \), say \( x_0 > \xi \). Derive a contradiction with (ii).

For existence use (ii) and the fact that by the uniqueness that you just proved, (7) holds. Use the consequence of (iii) to show the existence.

For the approach to the boundary of \( D \) assume that \( G_+ \) is a compact subset of \( D \), and that \( \phi \) exists on a finite interval, either \( \xi \leq x < b \) or \( \xi \leq x \leq b \). Use either (iv) or (ii) and (v) to extend \( \phi \).

10. Let the functions \( k(x, t, y; \lambda), g(x; \lambda) \), and \( \alpha(\lambda) \) be twice continuously differentiable for \( a \leq x \leq b \), \( a \leq t \leq b \), all \( y \in \mathbb{R}^n \), and \( \lambda \in K^0 \), where \( K \) is a compact subset of \( \mathbb{R}^m \) and \( K^0 \) is its interior. Moreover, let \( a \leq \alpha(\lambda) \leq b \) on \( K \). If all the first and second derivatives of the functions \( k(x, t, y; \lambda), g(x; \lambda) \), and \( \alpha(\lambda) \) are uniformly bounded in their respective domains, show that the solution of the integral equation

\[ y(x; \lambda) = g(x; \lambda) + \int_{\alpha(\lambda)}^x k(x, t, y(t); \lambda) \, dt \]

is continuously differentiable.

HINT: Take the formal derivative of this integral equation with respect to a component, say \( \lambda' \), of \( \lambda \), and consider the resulting pair of equations for \( y \) and \( v = \partial_{\lambda'} y \) on the product \( y - v \)-space. Show that this pair satisfies a Lipschitz condition in \( (y, v) \).

11. Let \( x = \phi(t, x_0) \) be a solution of the initial-value problem

\[ \dot{x} = f(x, t), \quad x(t_0) = x_0. \]

Show that, under the appropriate smoothness hypotheses, its derivative \( v = \partial_{x_0} \phi(t, x_0) \) satisfies the first variation equation and the initial condition

\[ \dot{v} = \partial_{x} f(x, t)v, \quad v(t_0) = I, \]
respectively, where \( I \) is the identity matrix.

12. If the right-hand side of the differential equation \( \dot{x} = f(x, t) \) is \( r \)-times continuously differentiable, show that its solution \( x = \phi(t, x_0) \) with \( \phi(t_0, x_0) = x_0 \) is \( r \)-times continuously differentiable with respect to \( t, t_0, \) and \( x_0 \).

HINT: For \( x_0 \), use induction on \( r \). For \( t \) and \( t_0 \) also use the integral equation.

13. A family of functions \( F = \{f\} \) is equicontinuous on an interval \( I \), if given any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(t_1) - f(t_2)| < \varepsilon \) whenever \( |t_1 - t_2| < \delta \) and \( f \in F \). Prove Arzelà-Ascoli’s lemma

**Theorem 1** Let \( I \) be a compact interval, and let \( F = \{f\} \) be an infinite, uniformly bounded, and equicontinuous family of functions. Then \( F \) contains a sequence \( \{f_n | n = 1, 2, \ldots\} \), which is uniformly convergent on \( I \).

HINT: Let \( \{r_k | k = 1, 2, \ldots\} \) be all the rationals in \( I \) enumerated in some order. Show that there exist functions \( f_{nk} \in F \) such that the sequence \( \{f_{nk}\} \) converges at \( r_1, \ldots, r_k \), and that you can take \( \{f_n = f_{nn}\} \) as the desired uniformly convergent subsequence.

14. Prove Peano’s existence theorem

**Theorem 2** Let the function \( f(t, x) \) be continuous and bounded on the strip defined by \( 0 \leq t \leq 1, -\infty < x < \infty \). Then there exists at least one continuously differentiable solution of the initial-value problem

\[
\dot{x} = f(t, x), \quad x(0) = x_0
\]

on the interval \( 0 \leq t \leq 1 \).

HINT: Fix \( n \). For \( i = 0, \ldots, n \) put \( t_i = i/n \). Let \( \phi_n \) be a continuous function on \( 0 \leq t \leq 1 \) such that \( \phi_n(0) = x_0 \),

\[
\dot{\phi}_n(t) = f(t, \phi_n(t)) \quad \text{if} \quad t_i < t < t_{i+1},
\]

and put

\[
\Delta_n(t) = \dot{\phi}_n(t) - f(t, \phi_n(t)),
\]

except at the points \( t_i \), where \( \Delta_n(t) = 0 \). Then

\[
\phi_n(t) = x_0 + \int_0^t [f(\tau, \phi_n(\tau)) + \Delta_n(\tau)] \, d\tau.
\]

Choose \( M \) so that \( f < M \). Verify the following assertions:
(a) |\phi_n| \leq M, |\Delta_n| \leq 2M, \Delta_n \text{ Riemann integrable, and } |\phi_n| \leq |x_0| + M = M_1, \text{ say, on } 0 \leq t \leq 1, \text{ for all } n.

(b) \{\phi_n\} \text{ is equicontinuous on } 0 \leq t \leq 1, \text{ since } |\dot{\phi}_n| \leq M.

(c) Some \{\phi_{n_k}\} \text{ converges to some } \phi, \text{ uniformly on } 0 \leq t \leq 1.

(d) Since \( f \) is uniformly continuous on the rectangle \( 0 \leq t \leq 1, |x| \leq M_1, \) $$f(t, \phi_{n_k}(t)) \to f(t, \phi(t))$$

uniformly on \( 0 \leq t \leq 1. \)

(e) \( \Delta_n(t) \to 0 \) uniformly on \( 0 \leq t \leq 1 \) since

$$\Delta_n(t) = f(t_i, \phi_n(t_i)) - f(t, \phi_n(t))$$

for \( t_i < t < t_{i+1}. \)

(f) Hence

$$\phi(t) = x_0 + \int_0^t f(\tau, \phi(\tau)) d\tau.$$  

This \( \phi \) is the solution of the given problem.

15. Let \( q = q_1 + iq_2 \) and \( p = p_1 + ip_2, \) with \( q_j, p_j \in \mathbb{R} \) for \( j = 1, 2, \) and \( i^2 = -1. \) Consider the complex Duffing equation

\[
\dot{q} = p, \quad \dot{p} = q \left( K - \frac{1}{2} |q|^2 \right),
\]  

where \( K \) is an arbitrary constant.

(i) Show that system (9) is Hamiltonian with the Hamiltonian function

$$H = \frac{1}{2} |p|^2 + \frac{1}{2} \left( K - \frac{1}{2} |q|^2 \right)^2,$$

and that equations (9) are of the form

$$\dot{q} = 2 \frac{\partial H}{\partial p^*}, \quad \dot{p} = -2 \frac{\partial H}{\partial q^*},$$

where * denotes complex conjugation.

(ii) Show that the function

$$J = \frac{1}{2i} (q^* p - qp^*) = q_1 p_2 - q_2 p_1$$

7
is a first integral of system (9). Show that the Hamiltonian $H$ is invariant under the group of transformations $\{g^\theta\}$ given by the solutions of the system

$$\frac{dq}{d\theta} = 2 \frac{\partial J}{\partial p^*} = iq, \quad \frac{dp}{d\theta} = -2 \frac{\partial J}{\partial q^*} = ip.$$  

Show that if $q(\theta = 0) = Q \in \mathbb{R}$ is the initial condition of the first equation in this system, then the initial condition of the second equation must be $p(\theta = 0) = P + iJ/Q$ for some $P \in \mathbb{R}$.

(iii) Show that the coordinate change

$$q = Qe^{i\theta}, \quad p = \left(P + i\frac{J}{Q}\right)e^{i\theta},$$

with $Q$, $P$ and $\theta \in \mathbb{R}$, brings the Hamiltonian $H$ into

$$\mathcal{H} = \frac{1}{2} \left( P^2 + \frac{J^2}{Q^2} \right) + \frac{1}{2} \left( K - \frac{1}{2} Q^2 \right)^2,$$

and equations (9) into

$$\dot{Q} = P, \quad \dot{P} = Q(K - \frac{1}{2} Q^2) - \frac{J^2}{Q^2}, \quad \dot{\theta} = \frac{J}{Q^2}, \quad \dot{J} = 0, \quad (10)$$

and that this system is Hamiltonian with the Hamiltonian function $\mathcal{H}$. Notice that system (10) is decoupled in the following sense: Since $J$ is constant, the first two equations are a planar system for the variables $Q$ and $P$. Once this system is solved, the angle $\theta$ can be calculated by quadrature (i.e., integration).

(iv) Sketch the phase portraits of the $Q - P$ system for positive and negative $K$, and zero and nonzero $J$. What geometric objects in the full $q - p$ space do the periodic orbits in the $Q - P$ plane correspond to?

(v) There are two separatrix loops in the $Q - P$ phase plane for $K > 0$ and $J = 0$. Find the solutions on these two loops. Show that, in the full $q - p$ phase space, these solutions correspond to the solutions

$$q = 2\sqrt{K} \text{sech}(\sqrt{K}t)e^{i\theta_0}, \quad p = -2K \text{sech}(\sqrt{K}t) \tanh(\sqrt{K}t)e^{i\theta_0},$$

with constant $0 \leq \theta_0 \leq 2\pi$. What geometric object do these solutions trace out?

Remark: The remarkable connections between equations (9), the integral $J$, the group $\{g^\theta\}$, and the transformation that leads to system (10) are not accidental. In fact, almost every multi-dimensional conservative mechanical system solved to date is solvable because
of additional symmetries, which, in turn, generate conserved quantities and make it possible to reduce the system to quadratures.

16. **Reduction of order:** Let \( y_1(x) \) be a nonzero solution of the second order linear equation

\[
y'' + p(x)y' + q(x)y = 0. \tag{11}
\]

Show that the ansatz \( y = y_1(x)u \) will lead you to an equation for \( u \) which can be solved by quadrature, and hence to the general solution of (11).

17. **Euler’s Equation** reads

\[
x^2y'' + a_1xy' + a_0y = 0. \tag{12}
\]

(i) Show that if \( y(x) \) is a solution of (12), then so is \( y(-x) \). Deduce that it is enough to consider solutions of (12) for \( x > 0 \).

(ii) Show that assuming \( y = x^r \) in (12) leads to a quadratic equation for \( r \). Find the form of two linearly independent solutions of (12) if the roots of this quadratic equation are real \( r_1 \neq r_2 \).

(iii) Find the form of two linearly independent solutions of (12) if the roots of the quadratic equation for the exponent \( r \) are complex conjugate \( r_1 = \lambda + i\mu, r_2 = \lambda - i\mu \).

(iv) Use reduction of order or the Wronskian to find the solution of (12) if the quadratic equation for the exponent \( r \) has two equal real roots.

(v) Euler’s equation (12) is a perfect counter-example used to show what all can go wrong with the solutions of a differential equations at points where the existence theorem does not hold. Where does Euler’s equation have such points?

(vi) How far can the solution of Euler’s equation with the initial condition \( y(x_0) = y_0 \), with \( x_0 > 0 \), be extended.

(vii) Consider the situations in which the exponents \( r_1 \) and \( r_2 \) corresponding to a particular Euler’s equation are: (a) \( r_1 = 1, r_2 = 2 \); (b) \( r_1 = 1, r_2 = -1 \); (c) \( r_1 = r_2 = 1 \); (d) \( r_1 = i, r_2 = -i \); (e) \( r_1 = 1 + i, r_2 = 1 - i \); (f) \( r_1 = -1 + i, r_2 = -1 - i \). Write down the solutions in each case, draw their graphs, and discuss their behavior near the origin. Also, explain what goes wrong with the initial-value problem \( y(0) = y_0 \) in each case.

18. Show that if \( A(t) \) is a continuous \( n \times n \) matrix, \( \Phi \) is a fundamental matrix of the system

\[
\dot{x} = A(t)x, \tag{13}
\]
and $C$ is a constant nonsingular matrix, then $\Phi C$ is another fundamental matrix of system (13). Show also that every fundamental matrix of (13) is of this form.

19. Let $A^\dagger$ be the adjoint (complex conjugate and transposed) matrix of $A$. The system

$$ \dot{x} = -A^\dagger(t)x $$

(14)

is the adjoint system to system (13). Show that if $\Phi$ is a fundamental matrix of (13), then $\Psi$ is a fundamental matrix of (14) if and only if $\Psi^\dagger\Phi = C$ where $C$ is a nonsingular constant matrix.

20. Consider the equation

$$ \ddot{x} + q(t)x = 0, $$

(15)

where $q(t)$ is a continuous function, in the phase plane $(x, y = \dot{x})$. By following the outline below, you will prove the Sturm oscillation and comparison theorems, which are useful for analyzing the Sturm-Liouville eigenvalue problem.

(i) Show that trajectories of (15) intersect the ray $\{x = 0, y > 0\}$ at points where $x$ is increasing and the ray $\{x = 0, y < 0\}$ at points where $x$ is decreasing along the trajectory.

(ii) Deduce from part (i) that for any two successive intersections of a solution with the $y$-axis one occurs with $y > 0$ and the other with $y < 0$.

Let $\phi$ be the polar angle measured \textit{clockwise} from the positive $y$-axis.

(iii) Use part (ii) to show that between any two successive intersections of a trajectory with the $y$-axis, the angle $\phi$ increases by $\pi$ along this trajectory.

(iv) Show the \textbf{Sturm Oscillation Theorem}: On the interval between two successive zeros of any solution of equation (15) there is a zero of any other solution.

HINT: Consider the polar angles $\phi = \alpha(t)$ and $\phi = \beta(t)$. For any two linearly independent solutions, $\alpha(t) \neq \beta(t)$ for all $t$. (Why?)

(v) Show

$$ \dot{\phi} = \frac{q(t)x^2 + y^2}{x^2 + y^2} $$

(vi) Show the \textbf{Sturm Comparison Theorem}: Consider two equations of the form (15)

$$ \ddot{x} + q(t)x = 0, $$

(16a)

$$ \ddot{x} + Q(t)x = 0, $$

(16b)
and assume that \( Q(t) \geq q(t) \). Then on the interval between any two successive zeros of any solution of (16a) there is a zero of any solution of (16b).

HINT: First assume that \( Q(t) > q(t) \). Then use the result of part (v) to show that the polar angle \( \phi = A(t) \) corresponding to a solution of (16b) always grows faster than the angle \( \phi = \alpha(t) \) corresponding to a solution of (16a).

(vii) Show that the distance between any two successive zeros of (15) is

- not larger than \( \pi/\omega \) if \( q(t) \geq \omega^2 \) for all \( t \),
- not smaller than \( \pi/\Omega \) if \( q(t) \leq \Omega^2 \) for all \( t \).

In particular, if \( q(t) \leq 0 \) for all \( t \), then no solution of (15) except the identically zero solution can have more than one zero.

21. Consider now the \textbf{Sturm-Liouville eigenvalue problem}:

\[
\ddot{x} + (q(t) + \lambda) x = 0, \quad 0 < x < l, \quad x(0) = x(l) = 0, \quad (17)
\]

where \( \lambda \) is a constant parameter. The values of \( \lambda \) for which solutions of (17) that are not identically zero exist are called the \textit{eigenvalues}. The corresponding solutions are called the \textit{eigenfunctions}.

(i) Show that for any function \( q(t) \) that is smooth on the interval \([0, l]\), (17) has an infinite set of eigenvalues. The corresponding eigenfunctions may have an arbitrarily large number of zeros on this interval.

HINT: Consider the solution of (17) with initial condition \( x(0) = 0, \dot{x}(0) = 1 \), and let \( \phi = \alpha(t, \lambda) \) be its polar angle. Use the Sturm Comparison Theorem to show that \( \alpha(l, \lambda) \to \infty \) as \( \lambda \to \infty \). Conclude that there exists an infinite set of eigenvalues \( \lambda_n \) for which \( \alpha(l, \lambda_n) = n\pi \).

(ii) Show that

\[
\lim_{n \to \infty} \frac{\lambda_n}{n^2} = \left( \frac{\pi}{l} \right)^2.
\]

22. Let \( x = \phi(t) \) be a periodic solution of the system

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f \in C^r(\mathbb{R}^n) \text{ for some } r \geq 1. \quad (18)
\]

Let \( \Sigma \) be any local, \((n-1)\)-dimensional surface, called the Poincaré section, transverse (i.e., non-tangent) to the orbit \( O \) of the solution \( \phi(t) \). Let \( P : \Sigma \to \Sigma \) be the Poincaré map, that
is, for any appropriate point \( p \in \Sigma \), \( P(p) \) is the first point in which the orbit through \( p \) intersects \( \Sigma \) again for some \( t > 0 \).

(a) Draw a sketch of the orbit \( O \), the Poincaré section \( \Sigma \), and the Poincaré map \( P \). Show that \( P \) is well defined and continuous on some small enough subset \( U \subset \Sigma \).

(b) Show that any two Poincaré maps \( P \) and \( P' \) are differentiably conjugate, that is, there exist a diffeomorphism \( \sigma(P, P') \) (i.e., both \( \sigma \) and its inverse \( \sigma^{-1} \) are continuously differentiable) such that \( P = \sigma(P, P') \circ P' \circ \sigma^{-1}(P, P') \). Just what is this \( \sigma(P, P') \)?

HINT: You can assume that the solution of a differential equation is as smooth as its right-hand side in time, initial conditions, and parameters.

(c) Consider the linearization
\[
\dot{u} = Df(\phi(t))u
\]
of equation (18) about the solution \( x = \phi(t) \). Show that \( (n - 1) \) Floquet multipliers of this linear system are equal to the eigenvalues of any Poincaré map \( P : \Sigma \rightarrow \Sigma \), linearized about the fixed point \( p_\circ \) on \( \Sigma \) that is the intersection of \( \Sigma \) with the periodic solution \( x = \phi(t) \). Show that the remaining multiplier equals 1. What is the eigenvector that corresponds to this last multiplier?

23. Consider the system
\[
\begin{align*}
\dot{A}_1 &= -i\beta_1 A_1 + i\frac{\mu}{4}|A_1|^2 A_1 - iA_1^* A_2, \quad (19a) \\
\dot{A}_2 &= -i\beta_2 A_2 + i\frac{\mu^2}{2}|A_2|^2 A_2 - i\frac{i}{2}A_1^2, \quad (19b)
\end{align*}
\]
where \( A_1 \) and \( A_2 \) are complex, \( i^2 = -1 \), \( ^* \) denotes the complex conjugate and \( | \cdot | \) denotes the complex modulus.

(i) Show that the system (19) possesses an invariant plane with \( A_1 = 0 \). Find the general solution \( A_2 = a(t) \) in this plane.

(ii) Linearize the system (19) about the solution \( a(t) \), that is, assume the ansatz
\[
(A_1(t), A_2(t)) = (u_1(t), a(t) + u_2(t)),
\]
substitute this ansatz into (19), and neglect higher powers of \( u_1(t) \) and \( u_2(t) \).

(iii) Show that a time-dependent coordinate change transforms the resulting linear system into a linear system with constant coefficients, and thus solve it.

(iv) Compute explicitly the Floquet multipliers corresponding to the periodic solutions of system (19) in the plane \( A_1 = 0 \).
(v) Exhibit explicitly the solutions of the linearization of system (19) around the periodic solutions in the plane $A_1 = 0$ in the form implied by the Floquet theorem, that is, $\psi(t)e^{Rt}$, where $\psi(t)$ is periodic and $R$ is a constant matrix.

24. Consider the system

$$\begin{align*}
\dot{a} &= [\lambda + A^a(a, b, \mu)a + A^b(a, b, \mu)b]a, \\
\dot{b} &= [-\lambda + B^a(a, b, \mu)a + B^b(a, b, \mu)b]b,
\end{align*}$$

(20a) (20b)
defined for $|a|, |b| < \delta$ for some positive $\delta$. Here, $\lambda > 0$ and $\mu$ are constant parameters. Assume that all the functions in (20) are uniformly bounded in all their arguments.

(a) Show that the lines $a = 0$ and $b = 0$ are both invariant under the flow of system (20), and that the origin $a = b = 0$ is an equilibrium point.

(b) Let $\kappa$ be any number satisfying $0 < \kappa < \lambda$, and let and $\delta$ be sufficiently small. Consider a solution starting at $t = 0$ with the initial conditions $a = a_0$, $b = \delta$, and let $T$ be the time that it takes this trajectory to reach one of the lines $a = \delta$ or $a = -\delta$. Show that there exist two constants $C_a$ and $C_b$ such that for any $0 < t < \tau < T$, we have

$$|a(\tau)| \geq C_a e^{\kappa(\tau-t)}|a(t)|, \quad |b(\tau)| \leq C_b e^{-\kappa(\tau-t)}|b(t)|.$$ 

Just how small must $\delta$ be?

HINT: Show the second inequality; the first is shown in the same way in backward time. To show the second inequality, first use the variation of constants formula to eliminate the linear part, then use the uniform boundedness of the functions that are left on the right-hand side.

25. (i) Show that two linearly independent power series solutions of Airy’s equation

$$w'' - zw = 0$$

about the point $z = 0$ are given by

$$w_1(z) = \sum_{n=0}^{\infty} \frac{z^{3n}}{3^{3n}n! \Gamma\left(n + \frac{2}{3}\right)}, \quad w_1(z) = \sum_{n=0}^{\infty} \frac{z^{3n+1}}{3^{2n}n! \Gamma\left(n + \frac{4}{3}\right)}.$$ 

Recall that

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x} \, dx$$

for $\alpha > 0$, and is defined by analytic continuation for all other values of $\alpha$, except negative integers, where it has simple poles.
(ii) Show that the Airy function

$$Ai(z) = \frac{1}{2\pi i} \int_{C} e^{z\zeta - \zeta^3/3} d\zeta,$$

where $C$ is the path starting at $\infty$ with the argument $-2\pi/3$ and ending at $\infty$ with the argument $2\pi/3$, satisfies Airy’s equation and express it in terms of these solutions. What you should get is

$$Ai(z) = \sum_{n=0}^{\infty} \frac{z^{3n}}{3^{2n+\frac{2}{3}} n! \Gamma(n + \frac{2}{3})} - \sum_{n=0}^{\infty} \frac{z^{3n+1}}{3^{2n+\frac{4}{3}} n! \Gamma(n + \frac{4}{3})}.$$

HINT: Express $Ai(0)$ and $Ai'(0)$ in terms of the two power-series solutions at $z = 0$. To compute these two values, first deform the integration path to run along the imaginary axis (why can you do that?), then change the variable so it runs along the real axis. In computing $Ai(0)$, split the resulting integral in two and change one of the two integrals to get the sum of the two integrals $\int_{0}^{\infty} e^{\pm it^3/3} dt$. Show that you can rotate the integration paths to become the half-rays emerging from the origin at the angle $\pm \pi/6$, which will give you real integrals. Do something similar for $Ai'(0)$. Express all these integrals in terms of the Gamma function, and, just before the end, use the well-known formula

$$\Gamma(\alpha) \Gamma(1 - \alpha) = \frac{\pi}{\sin \pi \alpha}.$$

26. Show that for non-integer values of $\nu$, two linearly independent solutions of Bessel’s equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

are given by $J_\nu(x)$ and $J_{-\nu}(x)$, where

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

Use the property that $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$ in the series for $J_\nu(x)$. If $\nu$ is an integer, use the fact that $1/\Gamma(-k) = 0$ for non-negative integer $k$ to show that $J_\nu(x) = (-1)^\nu J_{-\nu}(x)$, so that these two functions give only one linearly independent solution. Show that when $\nu$ is a non-negative integer, another independent solution is given by

$$J_\nu(x) = 2J_\nu(x) \log \frac{x}{2} - \sum_{k=0}^{\nu-1} \frac{(\nu - k - 1)!}{k!} \left(\frac{x}{2}\right)^{2k-\nu} - \left(\frac{x}{2}\right)^{\nu} \frac{1}{\nu!} \sum_{k=1}^{\nu} \frac{1}{k} - \sum_{k=1}^{\infty} \left[ \sum_{m=1}^{k+\nu} \frac{1}{m} + \sum_{m=1}^{k} \frac{1}{m} \right] \frac{(-1)^k}{k!(\nu + k)!} \left(\frac{x}{2}\right)^{2k+\nu}.$$
Remark: The standard second solution in this case is the limit of the Neumann function,

\[ Y_\nu(x) = \frac{J_\nu(x) \cos(\nu \pi) - J_{-\nu}(x)}{\sin(\nu \pi)}, \]

as \( \nu \) tends to the appropriate non-negative integer. It is given by the formula

\[ Y_\nu(x) = \frac{1}{\pi} [2 \gamma J_\nu(x) + J_\nu(x)], \]

where

\[ \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \]

is the well-known Euler-Mascheroni constant.

27. Show that the equation

\[ w'' - \left( 1 + \frac{1}{2z} \right) w' = 0 \]

has formal solutions of the form

\[ z^\alpha \sum_{n=-\infty}^{\infty} a_n z^n \]

for any \( \alpha \), but that only special \( \alpha \) give a convergent series and hence genuine solutions.

28. Use the transformation method on the hypergeometric equation to find two independent solutions which are functions of \( 1/z \) valid in \( |z| > 1 \).

29. (i) Find the restrictions on \( p(z) \) and \( q(z) \) at a regular singular point of the equation

\[ w'' + p(z)w' + q(z)w = 0 \]

if the exponents (i.e., roots of the indicial equation) are to be 0 and 1/2.

(ii) Find the most general form of the second-order equation with regular singular points at 0, 1, \( k \), \( \infty \) and exponents 0, 1/2 at 0, 1, \( k \) (but not \( \infty \)).

30. (a) Set \( \zeta = kz, b = k \) in the series solutions \( z^{-a} F(1/z) \) in Problem 28, and show how the convergence is lost as \( k \to \infty \).

(b) Using one of the identities, the second solution for \( |z| > 1 \) can be written

\[ z^{-b} \left( 1 - \frac{1}{z} \right)^{c-a-b} \frac{c-a-b}{F \left( c-a, 1-a, b-1-a; \frac{1}{z} \right)} \].
Carry out the same limiting process on this solution. The resulting series is an asymptotic expansion.

31. Lamé’s equation, which has for regular singular points at 0, 1, \( k \), and \( \infty \), can be taken in the form

\[
\frac{d^2w}{d\zeta^2} + \frac{1}{2} \left( \frac{1}{\zeta} + \frac{1}{\zeta - 1} + \frac{1}{\zeta - k} \right) \frac{dw}{d\zeta} + \frac{A + B\zeta}{\zeta(\zeta - 1)(\zeta - k)} w = 0.
\]

Find the confluent form as \( k \to \infty \), and show that the confluent form includes a transformation of Mathieu’s equation

\[
\frac{d^2w}{dz^2} + 4 \left( \alpha + \beta \cos^2 z \right) w = 0.
\]

32. Classify singular points of the equation

\[
\frac{d^2w}{dz^2} + \zeta \frac{dw}{dz} + \mu w = 0, \quad \mu = \text{constant},
\]

and find the first term of the asymptotic expansion for each of the two solutions as \( z \to \infty \).

33. Find an approximate solution of the equation

\[
\frac{d^2y}{dx^2} + 256e^{4x} y = 0 \quad \text{in} \quad x > 0,
\]

with initial conditions \( y(0) = 0, \ y'(0) = 1 \). Find the approximate position and magnitude of the first maximum of \( y(x) \).

34. For water waves propagating in two space dimensions \( (x, z) \) on water of depth \( h(x) \), an approximate equation for the height of the waves \( \eta(x, z, t) \) is

\[
\frac{\partial^2 \eta}{\partial t^2} = gh(x) \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial z^2} \right) + gh'(x) \frac{\partial \eta}{\partial x}.
\]

This applies very approximately to long waves on a beach; the shoreline is \( x = 0 \) and the depth \( h(x) \) increases out to sea. Consider waves propagating mainly along the shoreline (i.e., in the \( z \) direction) of the form

\[
\eta(x, z, t) = f(x)e^{ik(z - ct)},
\]

where \( k \) and \( c \) are constant. (a) Show via the WKB method that the waves can be trapped by a transition region at a value \( x \) offshore. (b) In the case \( h(x) = \alpha x \), find approximate solutions for \( f(x) \) on the two sides of the transition region and connect them.
35. Derive a transformation of variables to put the equation
\[ \frac{d^2y}{dx^2} + \left[ \lambda^2 q(x) + r(x) \right] y = 0 \]
into the form
\[ \frac{d^2\eta}{d\xi^2} + \left[ \lambda^2 \xi + R(\xi) \right] \eta = 0. \]
(This transformation is motivated by the idea of getting the solution close to the Airy function everywhere, not just at transition points.)

36. Consider a vector field \( f(x) = (f_1(x_1, x_2), f_2(x_1, x_2)) \) on \( \mathbb{R}^2 \). Its divergence \( \nabla \cdot f(x) \) is defined as
\[ \nabla \cdot f(x) = \frac{\partial f_1(x_1, x_2)}{\partial x_1} + \frac{\partial f_2(x_1, x_2)}{\partial x_2}. \]

Prove the following

**Theorem (Bendixson’s criterion)** If on a simply connected region \( D \subset \mathbb{R}^2 \), the divergence \( \nabla \cdot f(x) \) of the vector field \( f(x) \) does not vanish identically and does not change sign, then the differential equation \( \dot{x} = f(x) \) has no closed orbits in \( D \).

Recall that a simply connected region is one without holes.

HINT: Use Green’s theorem.

37. The Grönwall inequality: Let \( \phi, \psi, \) and \( \chi \) be real-valued continuous (or piecewise continuous) functions on a real \( t \)-interval \( I: a \leq t \leq b \). Let \( \chi(t) > 0 \) on \( I \), and suppose for \( t \in I \) that
\[ \phi(t) \leq \psi(t) + \int_a^t \chi(s)\phi(s) \, ds. \]

Prove that on \( I \)
\[ \phi(t) \leq \psi(t) + \int_a^t \chi(s)\psi(s) \exp \left( \int_s^t \chi(u) \, du \right) \, ds. \]

HINT: Let \( R(t) = \int_a^t \chi(s)\phi(s) \, ds \) and show that \( \dot{R} - \chi R \leq \chi \psi \).

38. Show the following

**Theorem:** Let
\[ \dot{x} = Ax + f(x), \quad (21) \]
where $A$ is a real constant matrix with the eigenvalues all having negative real parts. Let $f$ be real, continuous for small $x$ and

$$\lim_{\|x\| \to 0} \frac{\|f(x)\|}{\|x\|} = 0.$$ 

Then the identically zero solution is asymptotically stable.

HINT: Use variation of constants to derive an integral equation for the solution $\phi$ of (21) with given $\phi(0)$. The estimate

$$\|e^{At}\| \leq Ke^{-\sigma t}$$

which holds for some $K$, $\sigma > 0$, and all $t \geq 0$ (why?), implies an integral inequality for $\|\phi(t)\|$. Now, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|f(x)\| \leq \varepsilon \|x\|/K$ for $\|x\| < \delta$. Conclude that so long as $\|\phi(t)\| < \delta$, one must have

$$e^{\sigma t}\|\phi(t)\| \leq K\|\phi(0)\| + \varepsilon \int_0^t e^{\sigma s}\|\phi(s)\| \, ds.$$ 

Use the Grönwall inequality to conclude that

$$\|\phi(t)\| \leq K\|\phi(0)\|e^{-(\sigma - \varepsilon)t}.$$ 

Choosing appropriately small (how small?) $\varepsilon$ and $\|\phi(0)\|$ leads to the conclusion of the proof.

39. Consider the differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f(0) = 0.$$ 

Let $V(x)$ be a real, continuously differentiable function defined in a small neighborhood $U$ of $0 \in \mathbb{R}^n$.

Prove the following

**Theorem**  If

(i) $V(0) = 0$ and $V(x) > 0$ in $U - \{0\}$, and

(ii) $\dot{V}(x) \leq 0$ in $U - \{0\}$,

then $0$ is a stable equilibrium. Moreover, if

(iii) $\dot{V}(x) < 0$ in $U - \{0\}$,
then 0 is an asymptotically stable equilibrium.

Draw an appropriate figure for each of the two situations. The function $V(x)$ is known as the Lyapunov function.

HINT: Given $\varepsilon > 0$, let $k(\varepsilon) = \min_{\|x\| = \varepsilon} V(x)$. Show that $\delta(\varepsilon)$ required in the definition of stability is given by $\delta(\varepsilon) = \min_{\|x\| = \varepsilon} V(x)$. For the asymptotic stability, assume the contrary, namely, that for every $\varepsilon > 0$ there exist $\delta(\varepsilon)$, $\lambda(\varepsilon) > 0$ and a solution $x_\varepsilon(t)$ such that $x_\varepsilon(0) < \delta(\varepsilon)$ and that $x_\varepsilon(t) \geq \lambda(\varepsilon)$ for all $t > 0$. Use the fact that $\dot{V}(x_\varepsilon(t)) \leq -d(\varepsilon) < 0$ for some $d(\varepsilon)$ to derive a contradiction with the fact that $V(x_\varepsilon(t)) > 0$ for all $t > 0$.


Consider the map $(x, y) \mapsto (x_1, y_1)$ of $\mathbb{R}^2$, given by

$$x_1 = sx + F(x, y), \quad y_1 = s'y + \Phi(x, y),$$

where $s > 1 > s'$, and the functions $F(x, y)$ and $\Phi(x, y)$ are smooth at the origin and begin with quadratic terms there.

(i) Consider a curve segment $C$ that passes through the origin and is determined by a function $y(x)$, whose derivative is bounded from above by $\alpha$ and from below by $-\alpha$ for some $\alpha > 0$ for all small enough $x$. Show that, restricted to small enough $x$, the image $C_1$ of $C$ is of the same type.

(ii) Let $C'$ be a curve segment of the same type as $C$. Verify that $y(x) - y'(x) \to 0$ as $x \to 0$ and $|y(x) - y'(x)|/x < \mu$ for some $\mu > 0$.

Let then $C_1$ and $C'_1$ be the images of $C$ and $C'$, respectively, and let $\mu_1$ be the analog for $C_1$ and $C'_1$ of the number $\mu$ introduced in part (ii). In the next several steps, you will show that $\mu_1/\mu$ can be taken as close to $s'/s$ as we please if we restrict the domain to small enough $x$. To this end, let $Y$ and $Y'$ be the ordinates on $C$ and $C'$ corresponding to the same abscissa $X$, which we can assume positive with no loss of generality. Let $y_1$ and $y'_1$ be the corresponding ordinates on $C_1$ and $C'_1$, and let the preimages of the points $(X, y_1)$ and $(X, y'_1)$ be the points $(x, y)$ and $(x', y')$ on the curves $C$ and $C'$, respectively.

(iii) Choose any $\eta > 0$, and let $x$ be small enough. Show that the assumptions about $C$ and the functions $F$ and $\Phi$ imply the inequalities

$$|y| < \alpha x$$

and

$$|X - sx| < \eta(x + |y|),$$

19
and that therefore $x$ and $x'$ are both smaller than

$$\frac{X}{s - \eta(1 + \alpha)}.$$  

(iv) If $y_0'$ is the ordinate of the point on $C'_1$ with abscissa $x$, show that

$$|y_0' - y| < \mu x < \frac{\mu X}{s - \eta(1 + \alpha)},$$

$$|y' - y_0| < \alpha|x' - x|. \tag{26}$$

(v) Show that since the magnitudes of the derivatives of $F$ and $\Phi$ are smaller than $\eta$,

$$|x - x'| < \frac{\eta}{s - \eta}|y - y'|, \tag{27}$$

and that (25), (26), and (27) imply

$$|y - y'| < \frac{|y_0' - y|}{1 - \frac{\alpha\eta}{s - \eta}} < \frac{\mu X(s - \eta)}{[s - \eta(1 + \alpha)]^2}. \tag{28}$$

(vi) Deduce that

$$|y_1 - y_1' - s'(y - y')| < \eta(|x - x'| + |y - y'|), \tag{29}$$

so that indeed

$$|y_1 - y_1'| < \mu \left|\frac{s'}{s} + \varepsilon(\eta, \alpha)\right| X, \tag{30}$$

where $\varepsilon(\eta, \alpha) \to 0$ as $\eta \to 0$.

You will now deduce the existence of the unstable manifold of the origin.

(vii) Let $C' = C_1$. Then $C'_1 = C_2$, the image of $C$ under the second iterate of the map (22). If we denote the image of $C$ under the $n$-th iterate of the map (22) by $C_n$, then show that the difference between the ordinates of the points on the curves $C_n$ and $C_{n+1}$ with the abscissa $X$ are smaller than $\mu \sigma^n X$, where $\sigma = \mu_1/\mu$.

(viii) Deduce that the result of the previous paragraph implies the existence of a unique limiting curve $K = \lim_{n \to \infty} C_n$, which is independent of the choice of $C$, invariant under the map (22), and which passes through the origin, and is not tangent to the $y$ axis there. The $x$ coordinates of points on $K$ grow in absolute value under interations of the map (22).

(ix) Show that $K$ is tangent to the $x$ axis at the origin.
41. In class, we saw that the transcritical and pitchfork bifurcations only occur when special conditions are imposed on the system under consideration. To find out what happens when these conditions are relaxed, consider the vector fields

\[ \dot{x} = \varepsilon + \mu x \pm x^2 \quad \text{and} \quad \dot{x} = \varepsilon + \mu x \pm x^3, \]

where \( \mu \) and \( \varepsilon \) are parameters. For each vector field, plot three bifurcation diagrams in the \( \mu - x \) plane: one for \( \varepsilon > 0 \), one for \( \varepsilon = 0 \), and one for \( \varepsilon < 0 \). To sketch the equilibria, compute \( \mu \) in terms of \( x \) rather than the other way around.

42. Consider local bifurcations of the families of Hamiltonian systems with one degree of freedom:

\[ \dot{x} = p, \quad \dot{p} = f(x, \mu). \quad (31) \]

(a) Show that each such system can be derived from the Hamiltonian function

\[ H(x, p, \mu) = \frac{1}{2} p^2 + V(x, \mu) \]

via the formulas

\[ \dot{x} = \frac{\partial H(x, p, \mu)}{\partial p}, \quad \dot{p} = -\frac{\partial H(x, p, \mu)}{\partial x}, \]

where \( V(x, \mu) = -\int f(x, \mu) \, dx \).

(b) Show that all equilibria of system (31) lie on the \( x \)-axis, and that the eigenvalues at each equilibrium occur in pairs \( \pm \lambda \).

(c) Consider the following six functions for \( f(x, \mu) \):

\[ \mu \pm x^2, \quad \mu x \pm x^2, \quad \mu x \pm x^3. \]

Describe how the equilibria of the corresponding systems (31) undergo the Hamiltonian versions of the saddle-note, transcritical, and pitchfork bifurcations, respectively. In particular, what is the linearization matrix at the bifurcation point? Show that these bifurcations do not only create new equilibria or exchange their stability, but also create new families of periodic orbits and new separatrix curves, or make the equilibria exchange the separatrix loops that they possess and families of periodic orbits that encircle them. Present three phase portraits for each function \( f(x, \mu) \): one for \( \mu > 0 \), one for \( \mu = 0 \), and one for \( \mu < 0 \).

43. Consider the system

\[ \dot{E} = P - \alpha E, \quad \dot{P} = ED - \beta P, \quad \dot{D} = -EP - \gamma(D - \mu). \]
Determine all the equilibria of this system and all their bifurcation points. Classify the bifurcations according to their type: saddle-node, transcritical, pitchfork, or Hopf.

44. Consider the Lorenz system
\begin{align}
\dot{x} &= \sigma(y-x), \\
\dot{y} &= \rho x - y - xz, \\
\dot{z} &= -\beta z + xy,
\end{align}
where $\sigma$ and $\beta$ are fixed positive constants and $\rho$ is a parameter. The point $x = y = z = 0$ is always an equilibrium of this system. Show that it undergoes a bifurcation at $\rho = 1$. Let $\mu = \rho - 1$, add the equation $\dot{\mu} = 0$ to the system (32), and use the method for approximating center manifolds with parameters to analyze this bifurcation.

45. For iterated maps, formulate the center manifold theorem and related results that were given in class for vector fields. You may assume that the map has the form
\begin{align}
x_{n+1} &= Ax_n + f(x_n, y_n), \\
y_{n+1} &= By_n + g(x_n, y_n),
\end{align}
where the matrices $A$ and $B$ have eigenvalues with moduli equal to 1 and different from 1, respectively, and the functions $f$ and $g$ vanish at the origin together with their first derivatives. Show that if the center manifold is given by the equation $y = h(x)$, then the function $h$ satisfies the equation
\[
\mathcal{N}(h(x)) = h(Ax + f(x, h(x))) - Bh(x) - g(x, h(x)) = 0.
\]
Use this equation to approximate the center manifold and the dynamics on it for the mapping
\begin{align}
x_{n+1} &= x_n - 2(x_n + y_n)^3, \\
y_{n+1} &= \frac{1}{2}y_n + (x_n + y_n)^3.
\end{align}

46. Show that one possible second order normal form of the vector field
\begin{align}
\dot{x} &= y + \mathcal{O}(x^2 + y^2), \\
\dot{y} &= \mathcal{O}(x^2 + y^2)
\end{align}
is
\begin{align}
\dot{x} &= y + a_1 x^2 + \mathcal{O}(3), \\
\dot{y} &= a_2 x^2 + \mathcal{O}(3)
\end{align}
by completing the following outline:

(i) Write $z = x + iy, \bar{z} = x - iy$. Show that in terms of these variables, equation (33) becomes
\[
\dot{z} = \frac{1}{2i}(z - \bar{z}) + \mathcal{O}(|z|^2),
\]
with the equation for $\dot{\bar{z}}$ being its complex conjugate.
(ii) Show that the homological equation in this case reads
\[ \mathcal{L}(h(z, \bar{z})) \equiv (z - \bar{z})(D_z h_n(z, \bar{z}) + D_{\bar{z}} h_n(z, \bar{z})) - h_n(z, \bar{z}) + \overline{h_n(z, \bar{z})} = F_n(z, \bar{z}), \]
where \( F_n(z, \bar{z}) \) represents terms of order \( \mathcal{O}(|z|^n) \).

(iii) Let \( H_n = \text{span}\{z^k \bar{z}^{n-k} | k = 0, \ldots, n\} \). Compute the images of the basis vectors of \( H_n \) under the map \( \mathcal{L} \).

(iv) Let \( n = 2 \) in part (iii). Show that an appropriately chosen subspace of \( H_2 \) complementary to \( \mathcal{L}(H_2) \) is spanned by the polynomial \( (z + \bar{z})^2 \). Conclude that to \( \mathcal{O}(|z|^3) \), the normal form of equation (35) becomes
\[ \dot{z} = \frac{1}{2\ell}(z - \bar{z}) + \frac{a}{4}(z + \bar{z})^2 + \mathcal{O}(|z|^3), \]
where \( a = a_1 + ia_2 \). Rewriting this equation in terms of the real components \( x, y, a_1, \) and \( a_2 \), you should obtain equation (34).

47. Consider the forced van der Pol equation
\[ \ddot{x} + \varepsilon \omega \left( x^2 - 1 \right) \dot{x} + x = \varepsilon F \cos \omega t, \] (36)
with \( \varepsilon \ll 1, 1 - \omega^2 = \varepsilon \sigma, \) and \( \sigma = \mathcal{O}(1) \).

(i) Transform (36) into the new set of coordinates \( (u, v) \) by the formula
\[ x = u \cos \omega t + v \sin \omega t, \quad \dot{x} = -\omega u \sin \omega t + \omega v \cos \omega t. \]
Show that the system of equations for \( u \) and \( v \) is in the form suitable for applying the averaging method.

(ii) Average the \( u - v \) equations to obtain
\[ \dot{u} = \frac{\varepsilon}{2\omega} \left[ u - \sigma v - \frac{u}{4} (u^2 + v^2) \right], \quad \dot{v} = \frac{\varepsilon}{2\omega} \left[ \sigma u + v - \frac{v}{4} (u^2 + v^2) - F \right]. \]

(iii) Rescale the averaged equations as follows:
\[ t \rightarrow \frac{2\omega}{\varepsilon} t, \quad v \rightarrow 2v, \quad u \rightarrow 2u, \]
and let \( \gamma = F/2 \) to obtain
\[ \dot{u} = u - \sigma v - u \left( u^2 + v^2 \right), \quad \dot{v} = \sigma u + v - u \left( u^2 + v^2 \right) - \gamma. \] (37)
Figure 1: The bifurcation diagram for system (1). Some of the significant points are
A: \((\sigma, \gamma) = \left( \frac{1}{\sqrt{3}}, \sqrt{\frac{8}{27}} \right)\), O: \((\sigma, \gamma) = \left( \frac{1}{2}, \frac{1}{2} \right)\), and C: \((\sigma, \gamma) = (0, 0)\).

(iv) Transform (37) to polar coordinates, and show that it undergoes a saddle-node bifurcation on the curve
\[
\frac{\gamma^4}{4} - \frac{\gamma^2}{27}(1 + 9\sigma^2) + \frac{\sigma^2}{27}(1 + \sigma^2)^2 = 0.
\]
This is the curve DAC marked SN in Figure 1.

(v) Show that (37) undergoes a Hopf bifurcation on
\[
8\gamma^2 = 4\sigma^2 + 1, \quad |\sigma| > \frac{1}{2}.
\]
This is the curve OE marked HB in the Figure 1.

(vi) Show that (37) has a single equilibrium in regions I and III, a sink in I and a source in III. Show that in region II there are two sinks and a saddle, and in region IV there is a sink, a saddle and a source.

(vii) In Figure 1, consider the broken lines \(- - -\) crossing the curves OA, OD, AB, BE, and OB. Draw phase portraits representing the flow on and to each side of the indicated curve. In particular, in what regions does (37) have a limit cycle, and what is its stability type? Help yourself with the fact that at large distances from the origin, \((u^2 + v^2) < 0\).

(viii) Are there any inconsistencies in this bifurcation picture, provided no bifurcations other than the ones listed occur?
(ix) What do the equilibria of the averaged system (37) and their bifurcations signify for the original forced van der Pol equation (36).

48. Consider the mapping $P$ of the unit square into the plane shown schematically in the figure. Assume that this mapping is a diffeomorphism, and describe its invariant set $\Lambda_P$ and the orbit structure of $P$ on $\Lambda_P$. In particular, show that $P$ is topologically conjugate to a shift on three symbols.

49. Consider a two-sided homoclinic tangle such as the one shown in the figure. Describe how you can construct a Smale horseshoe map which is topologically conjugate to the shift on two symbols in such a way that the two symbols denote passages around the left-hand and right-hand portions of the tangle, respectively. In particular, make this statement precise and prove it.

50. Consider the equation describing the driven and damped pendulum

\[ \ddot{x} + \varepsilon \delta \dot{x} + \sin x = \varepsilon \sin \omega t, \tag{38} \]

where $\delta \geq 0$ and $0 \leq \varepsilon \ll 1$.

(i) Write equation (38) as a first order system. Show that the extended phase space of this system is the cartesian product of a torus and a real line. Also show that, for $\delta = 0$, this system is Hamiltonian and find its Hamiltonian function.

(ii) For $\varepsilon = 0$, (38) is a two-dimensional Hamiltonian system. Use this fact to plot its phase portrait, and find explicit solutions on its two separatrices. What is the physical meaning of the motion on these two separatrices?
(iii) Use the Melnikov method to compute that, for small $\varepsilon > 0$ and small enough $\delta \geq 0$, there are two symmetric homoclinic tangles in the phase space of equation (38). State and prove this result as precisely as you can.

(iv) Describe the physical consequences of the two homoclinic tangles that you found in part (iii) by means of an appropriately chosen Smale horseshoe map. Just what does the term “chaotic dynamics” mean in this case?

(v) Do you think that any of the “chaotic” orbits that you have found are stable and can thus be observed experimentally?

51. Consider the Poincaré return mapping $P : \Pi_0 \to \Pi_0$, with $\Pi_0$ being the appropriate portion of the $y = 0$ plane associated with the Shilnikov phenomenon discussed in class. If the vector field contains a parameter $\mu$ whose value is zero exactly when the Shilnikov saddle-focus connection exists, convince yourself that the mapping $P$ is given by the formula

$$
\begin{pmatrix}
  x \\
  z
\end{pmatrix} \mapsto \begin{pmatrix}
  x (\frac{z}{\varepsilon})^{\frac{\lambda}{\rho}} \left[ a \cos \left( \frac{\omega}{\lambda} \log \left( \frac{z}{\varepsilon} \right) \right) + b \sin \left( \frac{\omega}{\lambda} \log \left( \frac{z}{\varepsilon} \right) \right) \right] + e\mu + \bar{x} \\
  x (\frac{z}{\varepsilon})^{\frac{\lambda}{\rho}} \left[ c \cos \left( \frac{\omega}{\lambda} \log \left( \frac{z}{\varepsilon} \right) \right) + d \sin \left( \frac{\omega}{\lambda} \log \left( \frac{z}{\varepsilon} \right) \right) \right] + f\mu
\end{pmatrix}
$$

(39)

for some appropriate constants $a, \ldots, f$ and $\bar{x}$.

(a) Show that under an appropriate rescaling of the variables and parameters, the map (39) becomes

$$
\begin{pmatrix}
  x \\
  z
\end{pmatrix} \mapsto \begin{pmatrix}
  \alpha x z^\delta \cos \left( \xi \log z + \phi_1 \right) + e\mu + \bar{x} \\
  \beta x z^\delta \cos \left( \xi \log z + \phi_2 \right) + \mu
\end{pmatrix}
$$

(40)

(b) Show that, if we assume $|\alpha z^\delta| \ll 1$, solving for the fixed points of the mapping (40) amounts to solving the equation

$$z - \mu = (e\mu + \bar{x}) \beta z^\delta \cos \left( \xi \log z + \phi_2 \right)$$

(41)

(c) Investigate the solutions of equation (41) graphically and show the following facts: If $0 < \delta < 1$, then there are finitely many fixed points for both $\mu > 0$ and $\mu < 0$, and infinitely many for $\mu = 0$, and if $\delta > 1$, then there are no fixed points for $\mu \leq 0$ except for the homoclinic orbit itself at $z = \mu = 0$, and there is precisely one fixed point for $\mu > 0$. Graph the dependence of the periods of the periodic orbits in the $x - y - z$ phase space that correspond to these fixed points of the map $P$ on the value of the parameter $\mu$. Note that the closer $z$ is to $z = 0$ at one of these fixed points, the longer the period, since $z = 0$ corresponds to a homoclinic orbit.

(d) Find the stability type of the fixed points of the Poincaré return map (40) that you found in part (c). Distinguish the cases $\delta > 1/2$ and $\delta < 1/2$. 

26
(e) When the Shilnikov homoclinic orbit is broken, the unstable manifold of the origin intersects the section $\Pi_0$ at the point $(e\mu + \bar{x}, \mu)$. Convince yourself that if the $z$ component of the image of this point is zero, you will obtain a new homoclinic orbit which passes once through a neighborhood of the origin before falling back into the origin. Show that this condition is given by the equation

$$(e\mu + \bar{x}) \beta \mu^\delta \cos (\xi \log \mu + \phi_2) + \mu = 0. \quad (42)$$

Show, therefore, that there are infinitely many values of the parameter $\mu$ for which such double-pulse orbits exist.

**HINT:** You may want to help yourself by looking at the papers by


52. Let

$$u(x) = \begin{cases} 0, & |x| > \xi, \\ -U, & |x| < \xi, \end{cases}$$

where $U$ and $\xi$ are positive constants.

(a) Consider the equation

$$-\frac{d^2 f}{dx^2} + u(x)f = k^2 f, \quad -\infty < x < \infty.$$ 

Compute the functions $f = \phi(x, k)$ and $f = \psi(x, k)$, with the asymptotic behavior $\phi(x \to -\infty, k) \to e^{-ikx}$ and $\psi(x \to \infty, k) \to e^{-ikx}$, that we have discussed in class. Find the coefficients $a(k)$ and $b(k)$ in the expansion

$$\phi(x, k) = a(k)\psi(x, k) + b(k)\bar{\psi}(x, k).$$

Compute the reflection coefficient $r(k)$. Verify explicitly that $a(k)$ is analytic in the half-plane $\text{Im} k > 0$. Find its zeros $\{i\kappa_n\}$, and the corresponding eigenfunctions $\{\phi(x, i\kappa_n)\}$. From the formula

$$\phi(x \to -\infty, i\kappa_n) \to b_ne^{-\kappa_n x}$$

find the coefficients $b_n$.

(b) Reconstruct the potential $u(x)$ from the spectral data $r(k)$, $\kappa_n$, and $b_n$. 

27
53. A linear triatomic molecule is simulated by a configuration of masses and ideal springs that looks like the following diagram:

The equilibrium length of the springs is $b$. Find the eigenfrequencies and normal modes for longitudinal vibration. Describe the physical meaning of the modes.

54. (a) Show that the differential equation

$$F\left(\frac{d^n w}{dz^n}, \frac{d^{n-1} w}{dz^{n-1}}, \ldots, \frac{d w}{dz}, w\right) = 0$$

(note that $z$ does not appear explicitly) can be reduced to an $(n-1)$th order equation by taking $u = dw/dz$ as a function of $w$.

(b) Find the solution of the equation

$$(2 - w) \frac{d^2 w}{dz^2} - \left(\frac{dw}{dz}\right)^2 = w - 2w^2$$

in terms of a function defined by an integral. That is, $z = \int g(w) \, dw$, where $g(w)$ is a known function.

55. If an $n$-th order differential equation for $w(z)$ is invariant under the transformation

$$z = \lambda^\alpha Z, \quad w = \lambda^\beta W$$

for all values of the parameter $\lambda$ and for certain constants $\alpha$ and $\beta$, show that it can be reduced to an $(n-1)$th order equation for $u(\zeta)$, where $u$ and $\zeta$ are new variables defined by

(i) $w = z^\gamma \zeta$, \quad $\frac{dw}{dz} = z^{\gamma-1} u$, \quad $\gamma = \frac{\beta}{\alpha}$, \quad if $\alpha \neq 0$,

(ii) $z = \zeta$, \quad $\frac{dw}{dz} = wu$, \quad if $\alpha = 0$. 

28
Use

\[ z^3 w'' + 2zw' - 2w^2 = 0, \]
\[ z^2 w'' + z^2 w' + zw' + w = 0, \]
\[ w'' + p(z)w' + q(z)w = 0, \]
\[ w'' + ww' + zw^4 = 0, \]
\[ w'' = \frac{w^{3/2}}{z^{1/2}}, \]

as illustrative examples (and if necessary to stimulate general proofs!).