1. Reduce the following numbers to the form $a + ib$:

(i) \( \frac{5i}{(1-i)(2-i)(3-i)} \),

(ii) \( \left( \frac{2+i}{3-2i} \right)^2 \),

(iii) \( \frac{z-1}{z+1} \), where $z = x + iy$, with $x$ and $y$ real,

(iii) \( \frac{1}{z^2} \), where $z = x + iy$, with $x$ and $y$ real.

2. (i) If $z$ is a complex number such that $|z| = 1$, compute $|1+z|^2 + |1-z|^2$.

(ii) For complex $a$ and $b$, calculate that

\[
|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2),
\]

and interpret this result geometrically.

3. If $z$, $a$, $b$, and $c$ are complex numbers, show that, geometrically, the equation $az + b\bar{z} + c = 0$ may represent a straight line or a single point, and give the conditions on the coefficients $a$, $b$, and $c$ for each case to occur. Are there any other possibilities?
4. Find all the solutions of the equation $z^3 = 1$. Write the real and imaginary parts of these roots in terms of fractions involving integers and square roots of integers.

5. If $a$ and $b$ are real numbers, $b \neq 0$, show that

$$
\sqrt{a + ib} = \pm \left( \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + i \frac{b}{|b|} \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}} \right),
$$

and explain why the expressions under the square root signs are non-negative.

HINT: Assume $(x + iy)^2 = a + ib$, and find two equations for $x$ and $y$. From these equations, deduce $(x^2 + y^2)^2 = a^2 + b^2$, and then deduce the expressions for $x^2$ and $y^2$. Finally, be careful about choosing the relative signs of $x$ and $y$.

6. Use the result of problem 5 (even if you did not derive it), to compute

(i) $\sqrt{1 + i}$,

(ii) $\sqrt{\frac{1 - \sqrt{3}i}{2}}$.

7. If $n$ is a positive integer, and

$$
\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},
$$

compute that

$$
1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = 0
$$

for any integer $h$ which is not a multiple of $n$. What is the geometric interpretation of this equality? What happens if $h$ is a multiple of $n$?

HINT: For the geometric representation, dividing the expression by $n$ may be helpful.

8. Let $z = x + iy$, with $x$ and $y$ real.

(i) Write the function $z^3 - 2z^2 + z - 1$ in the form $u(x, y) + iv(x, y)$, and verify that $u$ and $v$ satisfy the Cauchy-Riemann equations $u_x = v_y$, $v_x = -u_y$ and also Laplace’s equation $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$.

(ii) Write the function $z\bar{z} + \bar{z} - 1$ in the form $u(x, y) + iv(x, y)$, and verify that $u$ and $v$ do not satisfy the Cauchy-Riemann equations $u_x = v_y$, $v_x = -u_y$. Does either $u$ or $v$ satisfy the Laplace equation $u_{xx} + u_{yy} = 0$ or $v_{xx} + v_{yy} = 0$? What can you conclude from this?
9. Verify that the function \( u(x, y) = e^x \cos y \) is harmonic, i.e., satisfies Laplace’s equation \( u_{xx} + u_{yy} = 0 \). Then, find its harmonic conjugate, i.e., a function \( v(x, y) \) such that \( u_x = v_y, \) \( v_x = -u_y \)? Verify that \( v \) also satisfies Laplace’s equation, \( v_{xx} + v_{yy} = 0 \)?

10. Let \( z = x + iy \) and \( w = u + iv \), with \( x, y, u, \) and \( v \) real. Consider the mapping by the analytic function \( w = z^2 \).

(i) What regions in the \( z \)-plane are mapped into the strips \( 0 < u < a \) and \( -a < u < 0 \), with \( a > 0 \), in the \( w \)-plane? What about the strips \( b < u < c \), where \( b \) and \( c \) are real?

(ii) What regions in the \( z \)-plane are mapped into the strips \( 0 < v < a \) and \( -a < v < 0 \), in the \( w \)-plane? What about the strips \( b < v < c \)?

(iii) What region in the \( w \)-plane is the wedge \( z = re^{i\theta}, r > 0, 0 < \theta < \alpha \) mapped onto? What about the wedge \( -\beta < \theta < \alpha \)? Here \( \alpha, \beta > 0 \).

Make appropriate sketches of the regions involved in all these cases.

11. Solve the differential equation \( w' = w \) with the initial condition \( w(0) = 1 \) using a power series \( w = \sum_{n=0}^{\infty} c_n z^n \). What values of the coefficients do you get? What is the function \( w = f(z) \), represented by this series?

12. Let \( z = x + iy \) and \( w = u + iv \), with \( x, y, u, \) and \( v \) real. Consider the mapping by the analytic function \( w = e^z \). What regions in the \( w \)-plane are the rectangles \( a < x < b, \) \( c < y < d \), with \( a, b, c, \) and \( d \) real, mapped into? Make appropriate sketches of the regions involved in all these cases. Pay specific attention to limiting cases of strips when one or both of the sides drift off to infinity. Make appropriate sketches of the regions involved.

13. Show that \( \log e^z = z + 2n\pi i \), where \( n \) runs through all the integers.

14. Compute all the values of \( \log(-ei) \).

15. Show that, when restricting to values on the principal branch of the logarithm, defined by \( -\pi < \arg z < \pi \),

(i) \( \log(1 + i)^2 = 2\log(1 + i) \),

(ii) \( \log(-1 + i)^2 \neq 2\log(-1 + i) \).

16. Find all the values for \( (1 + i)^i \).
17. Define
\[ \cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}. \]

(i) Sketch \( \cosh x \) and \( \sinh x \) for real \( x \).

(ii) Derive the addition formulas for \( \cosh(z + w) \) and \( \sinh(z + w) \), and also the formula \( \cosh^2 z - \sinh^2 z = 1 \).

(iii) What kind of a curve does the parametrization \( \xi = \cosh t, \eta = \sinh t, -\infty < t < \infty \) represent in the \((\xi, \eta)\)-plane?

(iv) Find the derivatives of \( \cosh z \) and \( \sinh z \).

(v) Find the power series representations for \( \cosh z \) and \( \sinh z \).

(vi) Derive the formulas \( \cos z = \cosh iz \) and \( \sin z = -i \sinh iz \).

(vii) Derive the formulas for the inverse functions \( \cosh^{-1} z \) and \( \sinh^{-1} z \) in terms of logarithms, and then derive the formulas for their derivatives.

HINT: Ignore any question of branches and take all the square roots with the plus sign.

18. Using the polar representation, show that

(i) \((-1 + i)^7 = -8(1 + i)\).

(ii) \((1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)\).

19. (i) If \( w_0 \) is one of the cube roots of a nonzero complex number \( z_0 \), show that the other two cube roots are \( w_0 \epsilon \) and \( w_0 \epsilon^2 \), where \( \epsilon = e^{2\pi i/3} \).

(ii) Express \( \epsilon \) in the rectangular coordinates.

(iii) Let \( z_0 = 4\sqrt{2}(-1 + i) \). Verify that one of its roots is \( w_0 = \sqrt{2}(1 + i) \), and find the other two roots.

20. (i) Given complex \( \alpha \), show that the function
\[ f(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \]
where
\[ \binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}, n = 1, 2, \ldots, \]
where \( n! = n(n-1) \cdots 1 \), is analytic for \( |z| < 1 \).

HINT: Compute the radius of convergence of the power series representing it.

(ii) Compute that its derivative \( f'(z) \) equals \( \alpha f(z)/(1 + z) \).

(iii) Deduce that the derivative of \( (1 + z)^{-\alpha} f(z) \) vanishes, and therefore

\[
f(z) = (1 + z)^\alpha.
\]

(iv) Which branch of \( (1 + z)^\alpha \) does \( f(z) \) represent?

HINT: What is \( f(0) \)?

21. (i) Derive that, for \( |z| < 1 \),

\[
\frac{d}{dz} \left[ \log(1 + z) - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n} \right] = 0.
\]

Conclude that, again for \( |z| < 1 \),

\[
\log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}
\]

on the principal branch of \( \log z \).

(ii) Show that, if \( n \) is a positive integer and \( |z| \leq \frac{1}{2} n \),

\[
n \log \left( 1 + \frac{z}{n} \right) = z + f_n(z),
\]

where

\[
|f_n(z)| \leq \frac{|z|^2}{n}.
\]

(iii) Deduce that

\[
\left( 1 + \frac{z}{n} \right)^n \to e^z
\]

as \( n \to \infty \) for any complex \( z \).

22. Let \( C \) be the line segment \( x = t, y = t, 0 < t < 1 \) in the complex plane. Let \( f(z) = u(x,y) + iv(x,y) = z^2 \). Compute the following integrals:

(i) \( \int_C f(z) \, dz \),
(ii) \( \int_C u(x, y) \, dx + v(x, y) \, dy, \)

(iii) \( \int_C f(z) \, |dz|, \) where \( |dz|^2 = dx^2 + dy^2. \)

23. Compute \( \oint_{|z|=r} x \, dz \) over the circle traversed counter-clockwise in two ways. First, using a suitable parametrization, and second, by observing that \( x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2} \left( z + \frac{r^2}{z} \right) \) on the circle.

24. Let \( f(z) \) be analytic in a region in which the closed curve \( C \) lies. Show that \( \int_C f(z) f'(z) \, dz \) is purely imaginary.

HINT: You can use the Cauchy-Riemann equations to show that the real part of the integrand is an exact differential. Alternatively, you can assume that higher-order partial derivatives of the real and imaginary parts of \( f(z) \) exist, and use the Cauchy-Riemann equations and Green’s formula to show that the real part of the integral vanishes.

25. Let \( C \) be the circle \( |z - z_0| = r \) traversed counter-clockwise, and let \( \alpha \) be any nonzero real number. Parametrize \( C \) by \( z = z_0 + re^{i\theta} \), with \(-\pi < \theta < \pi\), and compute that

\[
\oint_C (z - z_0)^{\alpha-1} \, dz = 2iR^\alpha \frac{\sin(\pi\alpha)}{\alpha}
\]

on the principal branch of the integrand. What does this show for \( \alpha = n \), integer? In particular, can you deduce the value for \( \alpha = 0? \)

26. Use Green’s formula to show that if \( C \) is a positively oriented simple closed contour, then the area of the region enclosed by \( C \) can be calculated as

\[
\frac{1}{2i} \oint_C \bar{z} \, dz.
\]

27. Use the method described below to derive the integration formula

\[
\int_0^\infty e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).
\]
(i) Let \( C \) be the circumference of the rectangle with the vertices \(-a, a, a + ib, \) and \(-a + ib, a > 0,\) traversed counter-clockwise. Show that the sum of the integrals of \( e^{-z^2} \) along the lower and upper horizontal legs of \( C \) can be written as

\[
2 \int_{0}^{a} e^{-x^2} \, dx - 2eb^2 \int_{0}^{a} e^{-x^2} \cos 2bx \, dx,
\]

and that the sum of the integrals along the vertical legs on the right and left can be written as

\[
ie^{-a^2} \int_{0}^{b} e^{y^2} \, dy - ie^{-a^2} \int_{0}^{b} e^{y^2+2iy} \, dy.
\]

Thus, with the aid of the Cauchy’s theorem, show that

\[
\int_{0}^{a} e^{-x^2} \cos 2bx \, dx = e^{-b^2} \int_{0}^{a} e^{-x^2} \, dx + e^{-(a^2+b^2)} \int_{0}^{b} e^{y^2} \sin 2ay \, dy
\]

(ii) Rewrite

\[
\left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy
\]

in polar coordinates, and thus show that

\[
\int_{0}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.
\]

(iii) Use the estimate

\[
\left| \int_{0}^{b} e^{y^2} \sin 2ay \, dy \right| < \int_{0}^{b} e^{y^2} \, dy,
\]

and let \( a \to \infty \) in the last formula in part (i) to deduce the desired integration formula.

28. Compute the integral

\[
\oint_C \frac{e^{-z^2}}{z^2} \, dz,
\]

where \( C \) is any positively-oriented simple closed contour surrounding the origin.

29. Let \( C \) be the boundary of the square whose sides lie along the lines \( x = \pm 2 \) and \( y = \pm 2,\) described in the positive sense. What is the value of

\[
\oint_C \frac{\cos z}{z(z^2 + 8)} \, dz?
\]
30. Let $C$ be a positively-oriented simple closed contour. Compute the value of

$$g(w) = \oint_C \frac{z^3 + 2z}{(z - w)^n} \, dz, \quad n = 1, 2, \ldots,$$

both when $w$ is inside and outside $C$.

31. Let

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n.$$

(i) Show that $P_n(z)$ is a polynomial of order $n$. These polynomials are called Legendre's polynomials.

(ii) Show that

$$P_n(z) = \frac{1}{2^{n+1} \pi i} \oint_C \frac{(s^2 - 1)^n}{(s - z)^{n+1}} \, ds, \quad n = 0, 1, 2, \ldots,$$

where $C$ is any positively-oriented simple closed contour surrounding the point $z$.

(iii) When $z = 1$, show that the integrand in (ii) can be written as $(s + 1)^n/(s - 1)$, and deduce that $P_n(1) = 1$. Likewise, calculate that $P_n(-1) = (-1)^n$, $n = 0, 1, 2, \ldots$.

32. Use various forms of Cauchy’s theorem and integral formula, as well as the binomial formula, to evaluate the integral

$$\oint_C \left( \frac{z + 1}{z} \right)^{2n} \frac{dz}{z},$$

where $C$ is the unit circle centered at the origin. Use this result to establish the real integral formula

$$\frac{1}{2\pi} \int_0^{2\pi} (\cos \theta)^{2n} \, d\theta = \frac{2^n n!}{4^n (n!)^2}.$$

HINT: The binomial formula is

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k! (n-k)!}.$$

33. Find an example showing that a function analytic in a region $\Omega$ can have its minimum modulus inside $\Omega$, provided the value of this minimum modulus is zero.
34. Let $f(z)$ be an entire function such that $|f(z)| \leq C|z|$, where $C$ is a positive constant. Deduce that $f(z) = Az$ for some complex constant $A$.

HINT: Use an appropriate Cauchy’s estimate from class to conclude that $f''(z) = 0$ everywhere in the plane. Note that the constant $M_R$ in Cauchy’s inequality is less than or equal to $C(|z_0| + R)$.

35. Find the Taylor expansions around the origin of the functions

(i) $\frac{\sin z}{z}$,

(ii) $z \cosh z^2$,

(iii) $\frac{z}{z^4 + 9}$,

and determine their circles of convergence.

36. Find the Taylor series for the function $e^z$ around $z = 1$. Where does it converge?

37. (i) Using the definition of its coefficients in terms of the derivatives of $f(z)$ at the center of its circle of convergence, derive the Taylor series around the origin for the function

$$f(z) = \log(1 + z),$$

where the principal branch of the logarithm is taken. What is its radius of convergence?

(ii) Use part (i) to find the Taylor series around the origin for the function

$$g(z) = \log \frac{1 + z}{1 - z},$$

where, again, the principal branch is considered. What is the radius of convergence of this series?

38. Find the Laurent series representations in powers of $z$ for the functions

(i) $\frac{\cos z}{z}$,

(ii) $z^4 \cosh \frac{1}{z^2}$.

Where do they converge?
39. Find all the possible Taylor and Laurent series representations in powers of \( z \) for the functions

(i) \( \frac{1}{(z - 1)(z - 3)} \),

(ii) \( \frac{z}{1 + z^2} \).

40. Find the Taylor series around the origin for the function \( \frac{1}{1 + z^2} \), and determine its radius of convergence. Then, deduce the series for

\[
\arctan z \quad \text{and} \quad \frac{\arctan z}{1 + z^2}.
\]

Where do these two series converge?

41. The Euler numbers \( E_n, \ n = 1, 2, \ldots \), are defined by the Taylor series representation

\[
\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.
\]

(i) Where does this series converge?

(ii) Show that \( E_{2n+1} = 0 \) for all \( n = 1, 2, \ldots \).

(iii) Derive the recursion formula

\[
\sum_{k=0}^{n} \binom{2n}{2k} E_k = \sum_{k=0}^{n} \binom{n}{k}^2 E_k = 0,
\]

where

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

42. Using Taylor and/or Laurent series expansions compute the following limits at removable singularities:

(i) \( \lim_{z \to 0} \frac{1}{z} \left[ (1 + z)^{\frac{1}{2}} - e \right] \),
\[ (ii) \lim_{z \to 0} \frac{e^{z^2} - \cosh z}{z^2}, \]

\[ (iii) \lim_{z \to 0} \left( \frac{\sin z}{z} \right)^{\frac{1}{z^2}}. \]

43. What types of singularities do the following functions have and at what points? Are they isolated? Compute the residues at those singularities if applicable.

(i) \( \frac{1}{z(e^z - 1)} \),

(ii) \( \frac{z}{z^4 - 1} \),

(iii) \( \sin \frac{1}{z^2} \),

(iv) \( \frac{\cos z}{z^7} \),

(v) \( \frac{e^z - 1}{(\sin z)^3} \),

(vi) \( \frac{\sqrt{z}}{1 - z} \).

44. Some of the Laurent series representations of the functions in Problem 39 contain infinitely many terms with negative powers of \( z \). Nevertheless, those functions clearly do not have any essential singularities. Explain the apparent contradiction.

45. Find the value of the integral

\[ \oint_C \frac{3z^2 + 2}{(z - 1)(z^2 + 9)} \, dz, \]

taken counterclockwise around the circle (i) \( |z - 2| = 2 \), (ii) \( |z| = 4 \).

46. Derive that if \( C \) is a positively oriented circle \( |z| = 8 \), then

\[ \frac{1}{2\pi i} \oint_C \frac{e^{az}}{\sinh z} \, dz = 1 - 2\cos \pi a + 2\cos 2\pi a. \]

47. Show that

\[ \oint_C \frac{dz}{(z^2 - 1)^2 + 3} = \frac{\pi}{2\sqrt{2}}. \]
where $C$ is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

48. Use residues to derive the following improper integral formulas:

(i) $\int_{0}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}$,

(ii) $\int_{0}^{\infty} \frac{x^2 dx}{x^6 + 1} = \frac{\pi}{6}$,

(iii) $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} \, dx = \frac{\pi}{a^2 - b^2} \left( \frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right), \quad a > b > 0$.

In (iii), also find the result in the limit $a \to b$ in two ways: by taking the limit on the right-hand side, and by taking the limit in the integrand and evaluating the integral when $a = b$.

49. Using residues, derive the formula

$$\int_{0}^{\infty} \frac{dx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}.$$  

HINT: Integrate around the positively oriented perimeter of the circular sector $0 \leq r \leq R$, $0 \leq \theta \leq \pi/3$. Sketch this sector first.

50. Using residues and Jordan’s lemma, derive the formula

$$\int_{0}^{\infty} \frac{x^3 \sin x \, dx}{x^4 + 16} = \frac{\pi}{2} e^{-\sqrt{2}} \cos \sqrt{2}.$$  

51. Evaluate the Fresnel integrals

$$\int_{0}^{\infty} \cos x^2 \, dx = \int_{0}^{\infty} \sin x^2 \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}},$$

by using the following method:

(i) Integrate the function $e^{iz^2}$ around the positively oriented perimeter of the circular sector $0 \leq r \leq R$, $0 \leq \theta \leq \pi/4$ (sketch this perimeter!) to show that

$$\int_{0}^{R} \cos x^2 \, dx = \frac{1}{\sqrt{2}} \int_{0}^{R} e^{-r^2} \, dr - \text{Re} \int_{C_R} e^{iz^2} \, dz.$$
and
\[ \int_{0}^{R} \sin x^2 \, dx = \frac{1}{\sqrt{2}} \int_{0}^{R} e^{-r^2} \, dr - \text{Im} \int_{C_R} e^{iz^2} \, dz \]
where \( C_R \) is the arc \( z = Re^{i\theta} \) with \( 0 \leq \theta \leq \pi/4 \).

(ii) Show that the value of the integral along the arc \( C_R \) in part (i) tends to zero as \( R \) tends to infinity by first obtaining the inequality

\[ \left| \int_{C_R} e^{iz^2} \, dz \right| \leq \frac{R}{2} \int_{0}^{\pi/2} e^{-R^2 \sin \phi} \, d\phi, \]

and then applying as slightly modified version of Jordan’s inequality.

(iii) Use the formula
\[ \int_{0}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}, \]
which you (should have) derived in Problem 27 (ii) to find the desired result.

52. By transforming the integral to a contour integral around the unit circle in the complex \( z \)-plane, derive the formula
\[ \int_{0}^{2\pi} \frac{d\theta}{2 + \cos^2 \theta} = \frac{2\pi}{\sqrt{6}}. \]

53. Show that
\[ \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} \frac{e^{px}}{1 + e^x} \, dx = \frac{\pi}{\sin \pi p}, \quad 0 < \text{Re} \, p < 1, \]
by integrating \( f(z) \) around the rectangle with vertices \( R, R + 2\pi i, -R + 2\pi i, -R \), using residues, and letting \( R \to \infty \). Make sure to show that the integrals along the vertical sides of the rectangle vanish as \( R \to \infty \).

54. (a) Compute the integral
\[ I(a) = \int_{-\infty}^{\infty} \frac{\cos x - \cos a}{x^2 - a^2} \, dx, \quad a \geq 0, \]
by carrying out the following steps:

(i) Carefully use Taylor series and the summation of the geometric progression to show that \( \cos z - \cos a = (z^2 - a^2)h(z, a) \), where \( h(z, a) \) is analytic in \( z \) at \( z = a \). Also, note that the integrand is \( O(1/x^2) \) as \( x \to \pm \infty \). This should show that the integral is well defined.
(ii) Consider the integral
\[ \oint_C f(z) \, dz = \oint_C \frac{e^{iz} - \cos a}{z^2 - a^2} \, dz \]
where \( C \) is the positively-oriented contour consisting of the following segments: the large semicircle \( C_R = \{ z = Re^{i\theta} \mid 0 \leq \theta \leq \pi \} \), small semicircles \( C_{\epsilon_1} = \{ z = -a + \epsilon_1 e^{i\theta} \mid \pi \geq \theta \geq 0 \} \) and \( C_{\epsilon_2} = \{ z = a + \epsilon_2 e^{i\theta} \mid \pi \geq \theta \geq 0 \} \), and the line segments \([ -R, -a - \epsilon_1 ] \), \([ -a + \epsilon_1, a - \epsilon_2 ] \), and \([ a + \epsilon_2, R ] \). (Sketch this contour!) Argue that this integral vanishes.

(iii) Argue that \( \oint_{C_R} f(z) \, dz \to 0 \) as \( R \to \infty \).

(iv) Calculate that
\[ \oint_{C_{\epsilon_j}} f(z) \, dz \to \frac{\pi \sin a}{2a}, \quad j = 1, 2, \]
as \( \epsilon_1, \epsilon_2 \to 0 \).

(v) Conclude that
\[ \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix} - \cos a}{x^2 - a^2} \, dx + \frac{\pi \sin a}{a} = 0, \]
where P.V. denotes the Cauchy principal value, and calculate \( I(a) \) by taking the real part of this equality.

(b) Let \( a \to 0 \) in \( I(a) \), and use a new variable \( t = x/2 \) to show that
\[ \int_0^{\infty} \left( \frac{\sin t}{t} \right)^2 \, dt = \frac{\pi}{2}. \]

55. (a) Show that
\[ \int_0^{\infty} \frac{\log x}{(x^2 + 4)^2} \, dx = \frac{\pi}{32} (\log 2 - 1) \]
by carrying out the following steps:

(i) Use residues to evaluate the integral of the branch
\[ f(z) = \frac{\log z}{(z^2 + 4)^2}, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \]
(i.e., defined in the complex plane cut along the negative imaginary axis) along the positively-oriented contour \( C \) consisting of the following segments: the large semicircle \( C_R = \{ z = Re^{i\theta} \mid 0 \leq \theta \leq \pi \} \), the small semicircle \( C_{\rho} = \{ z = \rho e^{i\theta} \mid \pi \geq \theta \geq 0 \} \), and the two intervals.
$L_\pm = [-R, -\rho] \text{ and } L_+ = [R, \rho]$. In particular, show that $f(z)$ has a second-order pole at $2i$, and evaluate that

$$\int_C f(z) \, dz = \frac{\pi}{16} (\log 2 - 1) + \frac{i\pi^2}{32}. \quad (1)$$

(ii) Show that

$$\int_{L_+ \cup L_-} f(x) \, dx = \int_\rho^R \frac{2\log r + i\pi}{(r^2 + 4)^2} \, dr$$

(iii) Carefully estimate that

$$\int_{C_\rho} f(z) \, dz = \mathcal{O}(\rho \log \rho) \to 0, \quad \text{as } \rho \to 0,$$

and

$$\int_{C_R} f(z) \, dz = \mathcal{O}(\log R/R^3) \to 0, \quad \text{as } R \to \infty,$$

and so conclude the validity of the desired formula by taking real parts in (1).

(b) As a byproduct, by taking imaginary parts in (1), show that

$$\int_0^\infty \frac{dx}{(x^2 + 4)^2} = \frac{\pi}{32}.$$ 

56. Derive the formula

$$\int_0^\infty \frac{dx}{(x + a)(x + b)} = \frac{\log b/a}{b - a}, \quad a > b > 0,$$

by carrying out the following steps:

(i) Integrate the branch of

$$f(z) = \frac{\log z}{(z + a)(z + b)}, \quad \text{with } 0 < \arg z < 2\pi,$$

(i.e., defined in the complex plane cut along the positive real axis) along the positively-oriented keyhole contour, $C$, consisting of the large circle $C_R = \{z = Re^{i\theta} \mid 0 < \theta < 2\pi\}$, small circle $C_\epsilon = \{z = \epsilon e^{i\theta} \mid 2\pi > \theta > 0\}$ and two segments of the real axis between $\epsilon$ and $R$, one along the top and the other along the bottom edge of the branch cut, taken in the opposite directions.

(ii) Carefully estimate that

$$\int_{C_\epsilon} f(z) \, dz = \mathcal{O}(\log \epsilon) \to 0, \quad \text{as } \epsilon \to 0,$$
and
\[ \int_{C_R} f(z) \, dz = O(\log R / R) \to 0, \quad \text{as } R \to \infty, \]

(iii) Show that the integral of \( f(z) \) along the bottom edge of the branch cut equals
\[ - \int_{\epsilon}^{R} \frac{\log x + 2\pi i}{(x + a)(x + b)} \, dx. \]

(iv) Compute the residues of \( f(z) \) at \( z = a \) and \( z = b \), and combine with the integrals along the top and bottom edges of the branch cut to derive the desired formula.

WARNING: Trivially splitting the integrand into partial fractions and using the real integration of the function \( 1/(x + \gamma) \) will bring you no points at all.

57. (a) Derive the partial fraction expansion
\[ f(z) = \frac{1}{\sin z} - \frac{1}{z} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n} \left( \frac{1}{z - n\pi} + \frac{1}{n\pi} \right) = z \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n\pi(z - n\pi)}, \quad (2) \]
by carrying out the following steps:

(i) Show that \( f(0) = 0 \).

(ii) Show that \( f(z) \) has simple poles at the points \( z = n\pi, n = \pm 1, \pm 2, \pm 3, \ldots \), with residues \( (-1)^n \).

(iii) Let \( C_n \) be the positively oriented boundary of the rectangle \(- (n + 1/2)\pi \leq x \leq (n + 1/2)\pi), -n\pi \leq y \leq n\pi, \) where \( z = x + iy \). Show that \( |\sin z| = O(e^{n|\pi|}) \) on the top and bottom, and so \( f(z) = O(1/n) \) there. Show also that \( |\sin z| = \cosh y \) on the sides, use this to bound \( |f(z)| \) by a constant there, and so bound \( |f(z)| \) by a constant along the entire \( C_n \).

(iv) Deduce formula (2) from the appropriate result derived in class.

(b) Combining terms in formula (2), conclude that
\[ \frac{1}{\sin z} - \frac{1}{z} = 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n n^2\pi^2}. \]

(c) From formula (2), deduce the formula
\[ \cot z \cosec z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z - n\pi)^2}. \]
(d) Integrate the two sides of the middle equality in (2) between 0 and \( z \), and exponentiate, to obtain

\[
\tan\left(\frac{z}{2}\right) = \frac{\prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{2n\pi}\right) e^{\frac{z}{2n\pi}}}{\prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{(2n+1)\pi}\right) e^{\frac{z}{2(2n+1)\pi}}}.
\]

HINT: To integrate \(1/\sin z\), first use the appropriate formula for half angles, and then multiply the numerator and denominator by \(\cos(\frac{z}{2})\). Also, note that, at the lower limit, the integral on the left-hand side does not vanish but instead contributes \(\log 2\), thus \(z/2\) in the denominator.

(e) Conclude from part (d) and the infinite-product expansion for \(\sin z\) given in class that the analogous product for \(\cos z\) equals

\[
\cos z = \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{(n + \frac{1}{2})\pi}\right) e^{\frac{z}{(n + \frac{1}{2})\pi}} = \prod_{n=0}^{\infty} \left(1 - \frac{z^2}{(n + \frac{1}{2})^2 \pi^2}\right).
\]

58. Using the infinite product representation

\[
\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)
\]

show that

\[
\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.
\]

59. Find the number of roots of the equation

\[
\phi(z) = \frac{z^2 - 4}{z^2 + 4} + \frac{2z^2 - 1}{z^2 + 6} = 0
\]

that lie inside the unit circle, \(\{|z| < 1\}\).

HINT: Clear the fractions, and let \(p(z) = (z^2 - 4)(z^2 + 6)\) and \(q(z) = (z^2 + 4)(1 - 2z^2)\). Show that on \(\{|z| = 1\}\), \(|p(z)| \geq 21\) and \(|q(z)| \leq 13\), i.e., \(|p(z)| > |q(z)|\). Use Rouché’s theorem to show that \(p(z)\) and \(p(z) - q(z)\) have an equal number of roots in \(\{|z| < 1\}\). Since \(p(z)\) has none there, conclude that neither has \(\phi(z)\).
60. Show that the Fourier transform of the function

\[ f(x) = \frac{1}{\cosh ax}, \quad a > 0, \]

equals

\[ F(k) = \frac{\pi}{a \cosh \left( \frac{\pi k}{2a} \right)}. \]

HINT: Integrate around the rectangle in the \( z \)-plane with the vertices at \( R, R + i\pi/a, \) \(-R + i\pi/a, \) and \(-R, \) and use residues. Show that the integrals on the two vertical sides vanish as \( R \to \infty. \)

61. Use Fourier transforms to solve the boundary-value problem

\[ \phi_{xx} + \phi_{yy} = 0, \quad \text{in } -\infty < x < \infty, \quad y > 0, \]

with

\[ \phi = f(x) \quad \text{as } y \to 0, \]

\[ \phi \to 0 \quad \text{as } y \to \infty, \]

by completing the following steps:

(i) Let

\[ \phi(x, y) = \int_{-\infty}^{\infty} \Phi(k, y) e^{-ikx} \, dk, \quad (3) \]

and derive the equation

\[ \Phi_{yy} - k^2 \Phi = 0. \quad (4) \]

(ii) Show that the solution of equation (4) consistent with the boundary condition as \( y \to \infty \) is

\[ \Phi(k, y) = C(k) e^{-|k|y}. \]

(iii) Use part (ii) and the boundary condition at \( y = 0 \) to derive that

\[ C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{ik\xi} \, d\xi. \]

(iv) From equation (3) and parts (ii) and (iii), derive that

\[ \phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, f(\xi) \int_{0}^{\infty} dk \, e^{-k[y+i(x-\xi)]} + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, f(\xi) \int_{0}^{\infty} dk \, e^{-k[y-i(x-\xi)]}, \]

18
and conclude that
\[ \phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yf(\xi)}{y^2 + (x - \xi)^2} d\xi. \]

62. Compute the Laplace transforms of the functions below, and use the inverse Laplace transform and residues to reconstruct the original functions:

(i) \( f(t) = e^{\gamma t}H(t) \),
(ii) \( f(t) = t^n H(t) \),
(iii) \( f(t) = t^n e^{\gamma t} H(t) \).

Here, \( n \) is a non-negative integer, and \( H(t) \) is the Heaviside function, whose value is 0 for \( t < 0 \) and 1 for \( t > 0 \).

63. (i) Use term-by-term integration to show that the Laplace transform of the function
\[ u(t) = \frac{\sin t}{t} H(t), \]
where \( H(t) \) is the Heaviside function, is \( U(s) = \arctan(1/s) \).

(ii) Use another term-by-term integration and either Cauchy’s integral formula or residues to show that the inverse Laplace transform of the function \( U(s) \) from part (i) is the original \( u(t) \).

HINT: In (i), first assume that \( \text{Re } s > 1 \). Proceed formally without regard to different branches. In truth, \( U(s) \) has a branch cut between \(-i\) and \( i\), and is analytic in \( \text{Re } s > 0 \). The expansion you will be using is valid for \(|s| > 1\).

64. Use the Laplace transform to solve the initial-value problem
\[ \ddot{u} - 2\dot{u} + 2u = 6e^{-t}, \quad u(0) = 0, \quad \dot{u}(0) = 1. \]

65. Use the Laplace transform to solve the wave equation
\[ u_{tt} = u_{xx}, \quad x > 0, \quad t > 0, \]
subject to the conditions
\[ u(x, 0) = u_t(x, 0) = 0, \quad x > 0, \]
\[ u(0, t) = f(t), \quad t \geq 0, \quad f(0) = 0, \]
\[ u(x \rightarrow \infty, t) \rightarrow 0, \quad t > 0. \]

CHECK: The solution is a traveling wave: \( f(t - x) \) if \( t \geq x \) and \( 0 \) if \( t < x \).

66. (i) Show that the transformation \( w = T(z) = 1/z \) maps the hyperbola \( x^2 - y^2 = 1 \) in the \( z \)-plane into the lemniscate \( \rho^2 = \cos 2\phi \) in the \( w = re^{i\phi} \)-plane. In particular, what branch of the hyperbola gets mapped onto what lobe of the lemniscate. Sketch both. In what direction is the particular lobe of the lemniscate traversed if the corresponding branch of the hyperbola is traversed from negative to positive values of \( y \)? Indicate this in the sketch.

HINT: Use the fact that
\[ x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}. \]

(ii) Onto what object does \( 1/z \) map the hyperbola \( x^2 - y^2 = -1 \)? Sketch the hyperbola and its image. As in (i), discuss the traversing directions, and include the results in the sketch.

67. Using the polar representation \( z = re^{i\theta} \), show that the transformation
\[ u + iv = w = z + \frac{1}{z} \]
maps circles \( \{|z| = r_0\} \) onto ellipses with parametric representations
\[ u = \left( r_0 + \frac{1}{r_0} \right) \cos \theta, \quad v = \left( r_0 - \frac{1}{r_0} \right) \sin \theta, \quad 0 \leq \theta \leq 2\pi, \]
and foci at the points \( w = \pm 2 \). Then show how it follows that this transformation maps the entire circle \( \{|z| = 1\} \) onto the segment \(-2 < u < 2\) of the \( u \)-axis and the domains inside and outside the circle onto the rest of the \( w \)-plane. If the circle \( \{|z| = r_0\} \) is traversed counter-clockwise, how does the direction in which the image ellipse is traversed depend on \( r_0 \)?

68. Derive that the general linear fractional transformation that maps

(i) 0 to 0 and 1 to 1 is
\[ w = \frac{z}{(1 - \lambda)z + \lambda}, \]
where \( \lambda \) is an arbitrary complex parameter;

(ii) 0 to 1 and 1 to 0 is
\[ w = \frac{z - 1}{\lambda z - 1} = \mu \frac{z - 1}{z - \mu}, \]

20
where $\lambda = 1/\mu$ is an arbitrary complex parameter;

To what point is $\infty$ mapped and what point is mapped to $\infty$ in each case?

69. Show that a linear fractional transformation $T$ of the form

$$T(z) = \lambda \frac{az - z}{1 - \bar{a}z}, \quad |\lambda| = 1, \quad |a| < 1,$$

maps the unit disc $\{ |z| < 1 \}$ onto itself by completing the following outline:

(i) Derive that

$$|T(z)|^2 = \frac{|a|^2 - 2 \text{Re}(\bar{a}z) + |z|^2}{1 - 2 \text{Re}(\bar{a}z) + |a|^2|z|^2},$$

and deduce that $|T(z)| < 1$ precisely when

$$(1 - |a|^2)(1 - |z|^2) > 0.$$  

Note that, when $|a| < 1$, this is true exactly for $|z| < 1$, so $T$ maps $\{ |z| < 1 \}$ into itself.

(ii) Compute that

$$T^{-1} (w) = \bar{\lambda} \frac{a\lambda - w}{1 - a\lambda w},$$

and that this is a map of the same type as $T$, so that $T^{-1}$ also maps $\{ |z| < 1 \}$ into itself. What parameters replace $\lambda$ and $a$ in $T^{-1}$ versus $T$?

(iii) Show that the map $T$ is onto by choosing a point $w_0$ in $\{ |z| < 1 \}$ and arguing from (i) and (ii) that $w_0 = T(z_0)$ for some point $z_0$ in $\{ |z| < 1 \}$.

70. (i) If we interpret a linear fractional transformation

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

as a transformation of the $z$-plane onto itself, we can ask if $T$ has any fixed points, i.e., points $z$ that satisfy the equation $T(z) = z$. Show that, except for the trivial cases ($c = 0$ and $a = d$, or $b = c = 0$), there exist two fixed points. If $(d - a)^2 + 4bc = 0$, show that these two points coincide.

(ii) If a linear fractional transformation $T(z)$ has two fixed points, say $\alpha$ and $\beta$, use the cross-ratio to show that $T(z)$ is equivalent to the equation

$$\frac{w - \alpha}{w - \beta} = \lambda \frac{z - \alpha}{z - \beta}.$$
where \( \lambda \neq 0 \) is a complex constant.

(iii) Use (ii) to reproduce the result of problem 68 (i).

71. Show that when a circle is transformed into a circle under the transformation \( w = 1/z \), the center of the original circle is never mapped onto the center of the image circle.

72. Show that the following functions \( f \) map the indicated regions \( \Omega \) into the unit circle.

(i) \( f(z) = e^{i\psi} \frac{z^{\pi/\alpha} - a}{z^{\pi/\alpha} - \bar{a}} \),

where \( \psi \in \mathbb{R} \), \( \text{Im } a > 0 \), and \( D \) is the wedge \( 0 < \arg z < \alpha \);

(ii) \( f(z) = e^{i\psi} \frac{e^z - a}{e^z - \bar{a}} \), where \( \psi \in \mathbb{R} \), \( \text{Im } a > 0 \), and \( D \) is the strip \( 0 < \text{Im } z < \pi \);

(iii) \( f(z) = e^{i\psi} \frac{e^{iz} - a}{e^{iz} - \bar{a}} \), where \( \psi \in \mathbb{R} \), \( \text{Im } a > 0 \), and \( D \) is the strip \( 0 < \text{Re } z < \pi \).

73. Let \( C_1 = \{|z - i/2| = 1/2\} \) and \( C_2 = \{|z - i/4| = 1/4\} \), and let \( D \) be the crescent-shaped region enclosed between \( C_1 \) and \( C_2 \). (Sketch it!) Show that the inversion \( \zeta = 1/z \) maps \( D \) onto the strip \( -2 < \text{Im } \zeta < -1 \), and the transformation \( w = e^{\pi \zeta} \) maps this strip onto the upper half plane. Use these results, in conjunction with results shown in class, to find a conformal transformation that maps \( D \) onto the unit disc.

74. (i) Find the steady temperature in the half-strip \( \{x > 0, 0 < y < a\} \) sketch it!), if the temperature on the bottom is 0 and on the top is \( T_0 \), while the side at \( x = 0 \) is perfectly insulated. Sketch the isotherms.

(ii) Find the steady temperature in a circular sector bounded by the lines \( \theta = 0 \) and \( \theta = \alpha \) and the arc \( r = a \) (sketch it!), if the temperature on the two lines is 0 and \( T_0 \), respectively, and the arc is perfectly insulated. Sketch the isotherms.

75. (i) Show that the transformation

\[
\zeta = \frac{z}{z - 1 - i}
\]

maps the lens-shaped intersection, \( D \), of the discs \( \{|z - 1| = 1\} \) and \( \{|z - i| = 1\} \) (sketch it, and show that the point \( 1 + i \) is one of its two vertices!) into the wedge \( 3\pi/4 < \arg \zeta < 5\pi/4 \). Also, determine which of the two circular arcs bounding \( D \) is mapped onto which of the half rays bounding the wedge. (Make a sketch!)
(ii) Use part (i) and reasoning similar to that in problems 72 (i) and 74 (ii) to find a mapping of $D$ onto the upper half $w$-plane, such that the upper of the two arcs bounding $D$ is mapped onto the negative real-$w$ axis and the lower onto the positive real-$w$ axis.

(iii) Use part (ii) to find the steady temperature distribution in $D$ if the temperature on the lower arc bounding $D$ equals 0 and the temperature on the upper arc bounding $D$ equals $T_0$. Sketch the corresponding isotherms.

76. The complex potential of a line charge $q$ along an infinite wire with the complex coordinates $z = z_0 = x_0 + iy_0$ equals

$$F(z) = \phi(x, y) + i\psi(x, y) = \frac{q}{2\pi} \log(z - z_0),$$

where the principal branch of the logarithm is understood.

NOTE: $z$ is not the third dimension here. This problem is two-dimensional in that there is no dependence on the third dimension in it.

(i) What are the equipotential lines and the lines of force for this potential in the $z$-plane? Sketch them!

(ii) What is the complex potential of two opposite line charges at the complex-conjugate locations $z_0$ and $\overline{z_0}$.

(iii) Show that the equipotential lines of the potential $\phi$ in (ii) are given by

$$(x - x_0)^2 + \left(y - y_0 \coth \frac{\pi \phi}{q}\right)^2 = y_0^2 \left[\sinh \frac{\pi \phi}{q}\right]^{-2}. \tag{5}$$

Sketch them!

(iv) Show that the lines of force in the $z$-plane of the potential $\phi$ in (ii) are given by

$$(x - x_0 + \mu)^2 + y^2 = y_0^2 + \mu^2, \quad \mu \in \mathbb{R}.$$ 

Sketch them!

(v) Show that the real axis is an equipotential line with the value 0 for the potential $\phi$ in part (ii). Conclude that $\phi$ can thus also be used to describe the potential of a line charge $q$ located at $z = z_0$, with $\text{Im} z_0 > 0$, above a perfectly conducting plane represented by the real axis.

HINT: Expand (5) for small values of $\phi$.

NOTE: In a perfectly conducting body, all charges equilibrate instantaneously, so every part of the body is at the same value of the potential.
77. Find the complex potential of the line charge $q$ located at a $z = z_0$ inside the perfectly conducting wedge $0 < \arg z < \alpha$, with $0 < \alpha < \pi$.

HINT: Map the wedge onto the upper half-plane, $\text{Im} \, \zeta > 0$, and the potential as

$$F(\zeta) = \kappa \log \frac{\zeta - \zeta_0}{\zeta - \zeta_0},$$

and determine the constant $\kappa$ so that

$$F(z_0 + w) \sim \frac{q}{2\pi} \log w + \text{const.}$$

for small values of $w$.

78. (i) Find the complex potential of the line charge $q$ located at a point $z = z_0$ inside the perfectly conducting cylinder whose base is the perimeter of the unit circle.

(ii) Show that the equipotential lines are the circles

$$(x - \lambda x_0)^2 + (y - \lambda y_0)^2 = 1 - \lambda, \quad 0 < \lambda < 1.$$ 

Sketch them!

HINT: Help yourself with the results of problem 69, and map $z_0$ to the origin while leaving $\{|z| = 1\}$ alone.

79. (i) Find the complex potential and the velocity of the fluid flow generated by two sources with volume flow $m$ located at the complex conjugate points $z_0$ and $\overline{z}_0$.

(ii) Show that the equipotential curves can be expressed as

$$x = x_0 \pm \sqrt{\sqrt{4y^2y_0^2 + C - y^2 - y_0^2}},$$

with $C > -4y_0^4$. Show that, for $C < y_0^4$, these curves are oval-shaped, encircling one of the two points $z_0$ and $\overline{z}_0$, and do not reach the real axis. Show that, for $C = y_0^4$, the equipotential curve is a figure eight that encircles both $z_0$ and $\overline{z}_0$ and is pinched at the point $z = x_0$. Show that, for $C > y_0^4$, the equipotential curves are again ovals, that they encircle both $z_0$ and $\overline{z}_0$, that they are the widest in $x$ at $y = \pm y_0$ and the narrowest at $y = 0$, and that they intersect the real axis under the right angle. Sketch or plot these curves!

HINT: You can help yourself by plotting these curves using Matlab or some other convenient plotting software.
(iii) Show that the streamlines are hyperbolae of the form

\[(x - x_0)^2 - 2C(x - x_0)y - y^2 = -y_0^2,\]

with the asymptotic directions \(c \pm \sqrt{c^2 + 1},\) and that each of them passes either through the point \(z_0\) or the point \(\overline{z_0}.\) Sketch them!

(iv) Show that the real axis is a streamline, and thus conclude that the same problem also describes the flow in the upper half-plane, \(y > 0,\) against a flat plate at \(y = 0,\) generated by the source at \(z = z_0.\) Show that \(z = x_0\) is a stagnation point for this problem. Also, show that the velocity in \(y > 0\) always has an upward component for large enough \(z,\) i.e., as \(|z| \to \infty.\) Finally, show that the speed (i.e., the magnitude of the velocity) decays as \(|z|\) increases. What is its rate of decay?

80. In problem 79, let \(z_0 = ia,\) \(a > 0.\) Show that, when you let \(a \to \infty,\) subtract a growing constant term from the complex potential, and rescale the volume flow \(m\) in some appropriate way, you obtain the flow towards a stagnation point at \(z = 0,\) discussed in class. (Recall that its complex potential is proportional to \(z^2.\)) Note that the speed grows as \(|z| \to \infty\) in this flow in contrast to the flow in problem 79. Interpret this in terms of mass conservation. Therefore, this is an example in which the limits \(z \to \infty\) and \(\text{Im} z_0 \to \infty\) clearly do not commute.