ODE Definitions

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I have defined and used a number of terms that are not in Edsberg’s book. This document collects them all in one place so that you do not have to go searching through Sauer, Holmes, or other texts to have a complete list.

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• Hadamard’s Well-Posedness: Hadamard defines a well-posed problem as one with:
  – A solution,
  – A unique solution, and
  – A unique solution with behavior that depends continuously on initial conditions.

In particular, we can only solve problems numerically that are well posed. In practice we need the eigenvalues of either the system or its linearization to have negative real parts.

• Amplification factor: The amplification factor of an ODE method is the value \( \rho(\lambda, \Delta t) \) such that, when we attempt to solve \( y' = \lambda y \) with the method with a time step of \( \Delta t \), we get
  \[
  w_{n+1} = \rho(\lambda, \Delta t) w_n = \rho(\lambda, \Delta t)^n w_0
  \]

Put another way, the amplification factor depends on the principal eigenvalue and the time step. For example, for Forward Euler, we get \( \rho(\lambda, \Delta t) = (1 + \Delta t \lambda) \).

• Zero Stable: We say that an ODE method is zero stable if the \( |\rho(\lambda, \Delta t)| < 1 \) for sufficiently small \( \Delta t \) and any \( \lambda \) with a negative real part. Put another way, we should be able to find a time step such that the method is stable for any well-posed problem.

• Absolutely Stable, or A-Stable: We say that an ODE method is A-stable if its amplification factor is bounded by \( |\rho(\lambda, \Delta t)| < 1 \) for all \( \Delta t \) when \( \text{Re}(\lambda) < 0 \). Put another way, the stability region should include the left half of the complex plane. Examples include the backward Euler and trapezoid methods.

• Stability Region: Given some amplification factor \( \rho(\lambda, \Delta t) \) of an ODE method, the stability region of the method is the set where \( |\rho| < 1 \). These tend to not have easy descriptions; the best way to draw them is with a contour plot.