Linear Algebra Definitions

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• We say that an equation is linear if it may be written as

$$\sum_{i=0}^{n} a_i x_i = b$$

where each $a_i$ is a specified constant and $x_i$ is an unknown. For example:

$$\sin(x_1) + x_2 = 3$$

is nonlinear but

$$x_1 + 4x_2 = 3$$

is linear.

• A system of linear equations is either consistent or inconsistent:
  - A system of linear equations is consistent if it has a solution (i.e., it will have either exactly one solution or infinitely many solutions)
  - A system of linear equations is inconsistent if it has no solution.

• There are three elementary row operations:
  1. Adding a multiple of one row to another. For example: if we add two times the first row to the second row and put the result in the second row then we write

$$2\text{row}_1 + \text{row}_2 \rightarrow \text{row}_2.$$

  2. Scaling a row by a nonzero constant. For example: if we want to scale the first row by two then we write

$$2\text{row}_1 \rightarrow \text{row}_1.$$

  3. Switching the positions of two rows. For example: if we want to switch the first and second rows then we write

$$\text{row}_1 \leftrightarrow \text{row}_2.$$

• The main diagonal of a matrix is the entries with equal row and column indices: for example, the main diagonal of the matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}$$

is 1, 2, 3, 4.
• The **pivots** of a matrix are the entries we use during elimination to convert other entries to zeros. An example: if we do elimination
\[
\begin{pmatrix}
2 & 2 & 1 & 2 \\
4 & 2 & 2 & 2 \\
0 & 2 & 3 & 2
\end{pmatrix} \sim \begin{pmatrix}
2 & 2 & 1 & 2 \\
0 & -2 & 0 & -2 \\
0 & 2 & 3 & 2
\end{pmatrix}
\]
then the 2 in the top left is a pivot and (if we kept eliminating) the \(-2\) in the second row and second column would also be used as a pivot. Pivots are never zero.

We also sometimes refer to the columns in which we did elimination as the **pivot columns**. The corresponding variable is called a **pivot variable**.

• The **free variables** are the variables that are not pivot variables. For example: if we have the linear system
\[
\begin{pmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
2 & 5 & 3
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
\]
and we form the augmented system and eliminate
\[
\begin{pmatrix}
1 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 \\
2 & 5 & 3 & 1
\end{pmatrix} \sim \begin{pmatrix}
1 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix} \sim \begin{pmatrix}
1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
since we did elimination in the first and second columns we have that \(x\) and \(y\) are pivot variables, while \(z\) is a free variable. This is reflected in the solution:
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
-2 + z \\
1 - z \\
z
\end{pmatrix} = \begin{pmatrix}
-2 \\
1 \\
-1
\end{pmatrix} + \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}z
\]
\(z\) can have any value and we have still found a solution to the linear system.

• A matrix is **upper triangular** if it is square and does not contain any nonzero entries below its main diagonal. Example:
\[
\begin{pmatrix}
2 & 1 & 2 & 0 \\
0 & 2 & 0 & 3 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 5
\end{pmatrix}
\]
is an upper triangular matrix.

• A matrix is in **upper echelon form** if:
  1. All zero rows are at the bottom of the matrix.
  2. Johnson et al require that the first nonzero entry in each row (starting from the left) is a 1, but most authors don’t: take this requirement as more of a suggestion.
  3. If the \(i+1\)th row contains nonzero entries, then the first nonzero is in a column to the right of the first nonzero in the \(i\)th row.

An example:
\[
\begin{pmatrix}
1 & 1 & 2 & 2 & 2 & 0 \\
0 & 0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Another way to think of this is that the first nonzero entry in each row is a pivot and we have eliminated everything under each pivot.
The general solution, or complete solution, to a system $A\vec{x} = \vec{b}$ is usually written as

$$\vec{x} = \vec{x}_p + \vec{x}_h$$

where $\vec{x}_p$, the particular solution, is some solution to $A\vec{x}_p = \vec{b}$ (it doesn’t have to be unique; any solution will do) and $\vec{x}_h$ is a solution to the homogeneous equation $A\vec{x}_h = \vec{0}$. An example: from above, we have that the system

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a solution given by the augmented system

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 5 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so $x$ and $y$ are pivot variables and $z$ is a free variable, so the solution to the homogeneous equation is

$$\vec{x}_h = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} z$$

for any $z$. Similarly, one particular solution to

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

is

$$\vec{x}_p = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

(i.e., the solution we get for $z = 0$) so the complete solution is

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{x}_p + \vec{x}_h = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} z.$$