Math 4100 Lecture 8

September 25, 2015

Administrative Details

Topics for Today

First Half
- (3.5) Independence, Basis, and Dimension Again
- (3.6) Dimensions of the Four Subspaces

Second Half
- (3.6) Dimensions of the Four Subspaces
- Test Review

First Half

Bases for Matrix Spaces

Linear Dependence: The Cheap Way

Why are $\mathbb{R}^4$ and the space of $2 \times 2$ real matrices isomorphic? Why is this process usually called *vectorization*?
Dimensions of Various Matrix Spaces

Determine the dimensions of the following subspaces of $n \times n$ matrices:

- Upper triangular matrices
- Diagonal matrices
- Symmetric matrices

Example

Find a basis for the space of $2 \times 3$ matrices whose nullspace contains $(2,1,1)$. 

2
Function Spaces

Overview

Strang does not go into much depth here, and for good reason: infinite dimensional linear algebra is usually called functional analysis, and things get much harder in the infinite case. The study of approximating infinite dimensional things in finite dimensions is strongly related to numerical analysis.

To keep things easy, only consider the case where the underlying vector space is the space of analytic functions (functions that are infinitely differentiable and equal to their Taylor series).

Example

Consider the second derivative acting on the space of analytic functions (so the second derivative is an operator): What is the null space of

\[ \frac{d^2}{dx^2} \]

Example

What is the particular solution to

\[ \frac{d^2y}{dx^2} = 1 \]

What is the complete solution?
Example (Problem 33)
Find a basis for the space of functions that satisfy
\[ xy' - y = 0. \]

Dimensions of the Four Subspaces
What four subspaces does Strang call fundamental? How can we easily read them off if we have a matrix in echelon form?
Example

Find the dimensions of the four subspaces for the echelon matrix

\[
\begin{pmatrix}
1 & 2 & 3 & 0 & 1 \\
0 & 0 & 0 & 1 & 4 \\
\end{pmatrix}
\]
Second Half

Example

Describe the four subspaces associated with

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]
Computing the Four Subspaces

Row space of $A$

Remember that we can relate a matrix $A$ to its echelon form $R$ by

$$EA = R \Rightarrow A = E^{-1}R.$$ 

$A$ has the same row space as $R$ (all we did was rearrange rows above) so we could choose the first $r$ rows of $R$ as a basis for the row space.

The column space of $A$

$A$ and $R$ usually do not have the same column space. As

$$A\vec{x} = \vec{0} \iff EA\vec{x} = E\vec{0} \iff R\vec{x} = \vec{0}$$
The null space of $A$
Elimination does not change the solutions.

This leads us to the rank nullity theorem:

The left nullspace of $A$
We can derive this from the previous result:

The Fundamental Theorem of Linear Algebra (Part 1)
Strang’s usage here is nonstandard, but if linear algebra were to have a commonly agreed upon fundamental theorem, this would probably be it:
Example (Problem 6)

Without elimination, find dimensions and bases for the four subspaces for

\[
A = \begin{pmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.
\]