Math 4100 Lecture 21

December 1, 2015

Administrative Details

Topics for Today

• (10.3) The Fourier, and Fast Fourier, transforms

The roots of unity, again

Overview

Recall, from last time, that the roots of unity (solutions to \( z^n - 1 = 0 \)) are evenly spaced (equal arc length) around the unit circle. For this section we will use \( w \) as a solution to this equation, so \( w, w^2, w^3, \ldots, w^n \) are all solutions to \( z^n - 1 = 0 \).

Euler’s Formula

The key idea in this section is that we can interpolate periodic functions very quickly. We say that a function is periodic when its value is invariant under some offset: that is, \( f(x) \) is a periodic function on \([0,1]\) if \( f(x) = f(x + 1) \).

Everything done here is in terms of \( \exp(in2\pi x) \), which is periodic on the interval \([0,1]\) due to the identity

\[
\exp(i2\pi x) = \cos(n2\pi x) + i\sin(n2\pi x).
\]

which is commonly called Euler’s formula.
Example (Problem 9)

If \( w = \exp(2\pi i/64) \), then \( w^2 \) and \( \sqrt{w} \) are among which roots of unity?

Example (Problem 10)

- Draw the sixth roots of 1 on the unit circle. Prove that they add to zero.
- What are the three cube roots of 1? Do they also add to zero?
The Fourier matrix

An Example

Consider the \( n = 4 \) Fourier matrix, which has values

\[
F = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & w & w^2 & w^3 \\
1 & w^2 & w^4 & w^6 \\
1 & w^3 & w^6 & w^9
\end{pmatrix}.
\]

Note that for this case the solutions to \( z^4 - 1 = 0 \) are \( i, -1, -i, \) and 1, so \( w = i \). Suppose that we multiply \( F \) by a vector \( \vec{c} \). What does this value represent?
Relation to Least Squares

Why is $F$ like the matrix we formed when we did least squares?

The Fast Fourier Transform

Overview

The Fourier matrix and its inverse allow for us to go from Fourier coefficients to function values or vice versa. However, matrix-vector multiplication is an $O(n^2)$ operation: by exploiting symmetry we can get this down to $O(n \log(n))$.

The Fast Fourier Transform, or FFT, relies on recursion: the algorithm we pick in this part relies on the prime decomposition of the size of the matrix. We will keep things simple and only work with powers of 2, so the Fourier matrix is $2^k \times 2^k$ for some integer $k$. 
Structure in the Fourier matrix

Consider $F_4$ again. Which parts of this matrix are duplicated?
Example (Problem 4)

All entries in the factorization of $F_6$ involve powers of $w_6$, which is the sixth root of unity:

$$F_6 = \begin{pmatrix} I & D \\ I & -D \end{pmatrix} \begin{pmatrix} F_3 & 0 \\ 0 & F_3 \end{pmatrix} P$$

write down these matrices in terms of $1$, $w_6$, $w_6^2$, and $w_3 = w_6^2$ in $F_3$. 