Math 4100 Lecture 16

November 10, 2015

Administrative Details

Topics for Today

First Half

• Test 2: postmortem
• (6.4) More on diagonalization
• (6.4) More on SPD matrices

Second Half

• (8.3) Markov matrices

First Half

Test 2
More on diagonalization

This is not a point that Strang emphasized, but it is important nonetheless. What is the link between the diagonalization of a matrix and its eigenvectors?

More on Positive Definite (SPD) Matrices

Example (6.5 C)

This one is worked in the book but it is important. Let \( F(x, y) \) be sufficiently smooth. Does \( F \) have a minimum at \((0, 0)\) if, at that point, \( \partial F/\partial x = 0 \) and \( \partial F/\partial y = 0 \)?
Example (Problem 10)
Which $3 \times 3$ symmetric matrix $A$ produces the quadratic
\[ \vec{x}^T A \vec{x} = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)? \]
Why is $A$ positive definite?

Example (Problem 20)
Give a quick reason why each of these statements is true:

- Every positive definite matrix is invertible.
- The only positive definite projection matrix is $P = I$.
- A diagonal matrix with positive diagonal entries is positive definite.
- A symmetric matrix with a positive determinant may not be positive definite.
Second Half

(8.3) Markov Matrices

Introduction

We say that a (real valued) matrix $A$ is a Markov matrix if:

- Each entry in $A$ is nonnegative.
- Each column of $A$ sums to 1.

This definition provides two immediate properties. Let $A$ be a Markov matrix. Then:

- The product of $A$ and any nonnegative vector (a vector with nonnegative entries) is a nonnegative vector.
- If the components of $\vec{u}$ sum to 1, then so do the components of $A\vec{u}$.

Eigenvalues of Markov Matrices

Amazingly (this is a very nice result) all Markov matrices have a maximum eigenvalue of 1. To prove this, start with the eigenvalues of $A - I$. 
A corollary: asymptotic behavior

What does the previous result imply about the value (for $\vec{u} \neq 0$ and a Markov matrix $A$)?

$$\lim_{k \to \infty} A^k \vec{u}?$$

Example (Problem 7)

Find the eigenvalues and eigenvectors of $A$. Explain why $A^k$ approaches $A^\infty$, where

$$A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \text{ and } A^\infty = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}.$$
Example (Problem 5)

Every year, 2% of young people become old and 3% of old people become dead (no births). Find the steady state for

\[
\begin{bmatrix}
\text{young} \\
\text{old} \\
\text{dead}
\end{bmatrix}_{k+1} = \begin{bmatrix}
0.98 & 0.00 & 0.00 \\
0.02 & 0.97 & 0.00 \\
0.00 & 0.03 & 1.00
\end{bmatrix}
\begin{bmatrix}
\text{young} \\
\text{old} \\
\text{dead}
\end{bmatrix}_k
\]

The Perron-Frobenius Theorem

If all entries in a matrix $A$ are positive, then all numbers in

\[ A\tilde{x} = \lambda_{\text{max}}\tilde{x} \]

are strictly positive.
Markov matrices and PageRank

How do Markov matrices relate to the original Google search algorithm?