Math 4100 Lecture 13

Administrative Details

Topics for Today

First Half
- (5.3) Applications of determinants: Cramer’s Rule, Inverses, and Volumes

Second Half
- (6.1) Introduction to Eigenvalues

First Half

Cramer’s Rule

Derivation

Consider the matrix product

\[
A \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}.
\]

This provides the value for \( x_1 \) in terms of determinants:
Example
Solve
\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]
with Cramer’s rule.

An Application: Inverses
We can use Cramer’s rule to calculate entries in $A^{-1}$ without elimination, giving us the following formula:
Calculating Areas and Volumes

Area of a Triangle

The determinant provides a very convenient way to calculate the area of a triangle given the coordinates of its vertices. The high level explanation is that determinants and areas have equivalent properties, so calculating the determinant gives the area.

Volume of a box

Let $Q$ be an orthogonal matrix. We now have two ways to prove that it has a determinant of 1:
Polar Coordinates

The area proof also explains why the change of coordinates formula works in calculus:

The Cross Product

The cross product is defined in terms of the determinant, which yields its important properties:

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$
- $\vec{u} \times \vec{u} = \vec{0}$
Second Half

Introduction

Definitions

A matrix tends to push things in preferred directions. These directions are the eigenvectors. The “amount” in each direction is the associated eigenvalue. Formally, these quantities are defined as the pair \((\lambda_i, \vec{v}_i)\) where

\[ A\vec{v}_i = \lambda_i \vec{v}_i. \]

Using properties of determinants, how does this lead to a simple way of calculating eigenvalues and eigenvectors?

Example

Describe the eigenvalues and eigenvectors of the 180 degree rotation matrix.
Example

Find the eigenvalues and eigenvectors of

\[ A = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}. \]

What are the eigenvectors and eigenvalues of \( A^{10} \)?

Example

For the same \( A \) as above, find

\[ A^{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]
Example
Find the eigenvectors and eigenvalues of the matrix which projects vectors in $\mathbb{R}^2$ onto the vector $(1, 1)$.

Example
Find the eigenvalues of the rotation matrix corresponding to a 90 degree rotation.