Math 4100 Lecture 12

October 20, 2015

Administrative Details

Topics for Today
First Half
  • (5.1) Properties of Determinants
Second Half
  • (5.2) Permutations and cofactors

First Half

Introduction
Chapter 5 is a short chapter describing some of the important properties of the determinant of a square matrix, usually notated as det(A) or detA. Note: it only makes sense to talk about determinants of square matrices, so I will, in this chapter, mean “square matrix” whenever I say “matrix”.

There are three formulas for calculating the determinant:
  • Multiply the pivots (times 1 or −1)
  • Add up n! terms, where A is an n × n matrix
  • Combine n smaller determinants

A Property Example
Let
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]
Take for granted that det(A) = ad − bc. If det(A) = 0, then what may we conclude about the rows of A?
Properties of the Determinant

We will skip the proofs of most of these for now. It is not hard to verify them by hand for $2 \times 2$ matrices. Strang numbers these as follows:

1. The determinant of the identity matrix is 1.
2. If two rows, or two columns, are switched, then the determinant changes sign.
3. The determinant is a linear function with respect to row operations on one row at a time.
4. If two rows of $A$ are equal, then $\det(A) = 0$.
5. Subtracting a multiple of one row from another does not change the determinant.
6. A matrix with a row of zeros has zero determinant.
7. If $A$ is triangular, then $\det(A)$ is the product of its diagonal values.
8. The determinant of a matrix is zero if and only if the matrix is singular.
9. $\det(AB) = \det(A)\det(B)$.
10. $\det(A^T) = \det(A)$.
Example

Which row changes show that \( \det(J_3) = -1 \) but \( \det(J_4) = 1 \), where

\[
J_3 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
J_4 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

Example

Use elimination to compute the determinant of \( A \) by the product of the pivots, where

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{pmatrix}
\]
Example

Determine whether the following statements are true or false:

1. If $A$ is singular then $AB$ is singular.
2. The determinant of $A$ is always equal to the product of its pivots.
3. $\det(A - B) = \det(A) - \det(B)$.
4. $AB$ and $BA$ have the same determinant.
Second Half

Permutations and Cofactors

Calculating Determinants from the $LU$ factorization

Find the determinant of

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 6 \\ 1 & 0 & 2 \end{pmatrix}$$

by finding the $PA = LU$ factorization. This is what Strang calls “the pivot formula”.

The “Big” Formula

This is the classic approach and is derived from the linearity rule in 5.1.
Determinants by Cofactors

This is the “classic” algorithm for computing determinants and is likely something you have seen before. To demonstrate, calculate the determinant of

\[
\begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & 1 & -1 & 0 \\
0 & 1 & 1 & -1 \\
0 & 0 & 1 & 1
\end{pmatrix}.
\]