Administrative Details

Topics for Today

First Half
- (4.3) More on least squares

Second Half
- (4.4) Orthogonal bases and Gram-Schmidt

First Half

Example
What condition on \((t_1, b_1), (t_2, b_2),\) and \((t_3, b_3)\) puts those three points on a straight line?
Example

Find the height of the horizontal line which best fits the points

\[(0, 0), (1, 8), (2, 8), \text{ and } (3, 20).\]

Why can’t this “linear” fit be exact? What are the four errors?

Second Half

Orthogonal Bases

From chapter three, we know that any set of three linearly independent vectors in \(\mathbb{R}^3\) also forms a basis for \(\mathbb{R}^3\). However, it is almost always most convenient to work in an orthogonal coordinate system (e.g., the standard basis).

What are orthonormal vectors? What are orthogonal matrices? How would they help us solve least squares problems?
Examples of Orthogonal Matrices

There are three very common types of orthogonal matrices:

1. permutation matrices,
2. rotation matrices, and
3. reflection matrices.

Important Properties

Let $Q$ be an orthogonal matrix. Then
\[\|Q\vec{x}\| = \|\vec{x}\|\]
and
\[(Q\vec{x}, Q\vec{y}) = (\vec{x}, \vec{y}).\]
**Projection with Q instead of A**

Our original application involved projection onto a subspace. What would the equations look like if we had an orthogonal basis described by the columns of $Q$ instead of the arbitrary basis described by the columns of $A$?

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**Orthogonalization by Gram Schmidt**

This is one, of many, orthogonalization algorithms. In fact, there are specific choices for rotations and reflections (Givens rotations and Householder reflections) that are superior when done on a computer. However, this one is theoretically useful, simpler, and occasionally the best choice on a computer.

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**Example**

Orthogonalize the basis

\[
\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]

by the Gram Schmidt method. (of course, this a basis for $\mathbb{R}^3$, but it is only an example of the algorithm.)
The QR factorization

Note the triangular structure of the Gram Schmidt algorithm:

- The first vector is taken as-is.
- The second vector is orthogonalized against the first.
- The third vector is orthogonalized against the first and second.

This resembles (but is not equal to) the LU factorization by the following relationship:

Essential Properties of the QR factorization

There are many variants on the QR factorization: I always consider the one where \( Q \) is an orthogonal matrix.

- The QR factorization is not, in general, guaranteed to be unique.
- The QR factorization exists for all \( m \times n \) matrices. If \( A \) is not square, then \( R \) will not be square.

Example

Find a QR factorization of

\[
A = \begin{pmatrix}
1 & 2 & 4 \\
0 & 0 & 5 \\
0 & 3 & 6
\end{pmatrix}
\]