1. Let $X =$ time-to-failure (TTF). Then

$$f_X(x) = 0.125e^{-0.125x}, \quad x > 0,$$
$$F_X(x) = 1 - e^{-0.125x}, \quad x > 0,$$

where $f_X$ and $F_X$ are the probability density function (pdf) and distribution function (df), respectively. Thus,

$$P(X \leq 1) = F_X(1) = 1 - e^{-0.125(1)} = 0.1175.$$

Thus, 11.75% fail within the one-year warranty! Looks like this is one of the very early calculators.

Let $S =$ profit. Then

$$S = \begin{cases} 25 & \text{if } X > 1, \\ -25 & \text{if } X \leq 1, \end{cases}$$

and so

$$E(S) = 25P(X > 1) + (-25)P(X \leq 1)$$
$$= 25(1 - 0.1175) - 25(0.1175)$$
$$= $19.125.$$

The effect is to reduce profit by $5.88 per unit.

B. As above, $P(X \leq 1) = F_X(1) = 1 - e^{-0.0125(1)} = 0.0124$. Now only 1.24% fail during warranty. The sale price is still $75, but now

$$S = \begin{cases} 20 & \text{if } X > 1, \\ -35 & \text{if } X \leq 1, \end{cases}$$

and so

$$E(S) = 20(1 - 0.0124) - 35(0.0124)$$
$$= $19.32.$$

**Conclusion:** Expected profit is greater even though manufacturing costs are higher! Sure – the difference is only 20 cents, but if you sell 10 million calculators, that’s 2 million dollars and customer satisfaction is much greater.

2.A. See the graph of the pdf in Part B.

$$P(X < 12) = \text{area under pdf to left of 12} = 0.125.$$
B. 

\[ f_x(x) \]

\[ 1 \]

\[ 2 \]

\[ 11.75 \]

\[ 12.25 \]

\[ 12.75 \]

C. For \( x < 11.75 \), \( F_X(x) = 0 \).

For \( 11.75 \leq x \leq 12.25 \),

\[
F_X(x) = \int_{-\infty}^{x} f_X(y) \, dy = 0 + \int_{11.75}^{x} 4(y - 11.75) \, dy = 2x^2 - 47x + 276.125.
\]

For \( x > 12.25 \),

\[
F_X(x) = \int_{-\infty}^{x} f_X(y) \, dy = 0 + \int_{11.75}^{12.25} 4(y - 11.75) \, dy + \int_{12.25}^{x} 4(12.75 - y) \, dy
\]

\[
= 0 + F_X(12.25) - F_X(11.75) + \int_{12.25}^{x} 4(12.75 - y) \, dy
\]

\[
= 0.5 - 2x^2 + 51x - 324.625 = 51x - 2x^2 - 324.125.
\]

For \( x > 12.75 \), \( F_X(x) = 1 \).

**Sketch:**

\[ F_X(x) \]

\[ 1.00 \]

\[ 0.75 \]

\[ 0.50 \]

\[ 0.25 \]

\[ 0.00 \]

\[ 11.50 \]

\[ 11.75 \]

\[ 12.00 \]

\[ 12.25 \]

\[ 12.50 \]

\[ 12.75 \]

\[ 25^{th} = 12.1 \]

\[ 75^{th} = 12.4 \]
D. See graph.

E. Let \( \xi_p \) denote the \( p \)-quantile. By definition of a quantile, to find \( \xi_p \), solve the equation \( F_X(\xi_p) = p \). From the graph, it is clear that the 25\(^{th} \) quantile is between 11.75 and 12.25 and the 75\(^{th}\) is between 12.25 and 12.75. Thus, solve the quadratic equations

\[
2x^2 - 47x + 276.125 = 0.25 \quad \text{and} \quad 51x - 2x^2 - 324.125 = 0.75
\]
to obtain \( \xi_{0.25} = 12.104 \) and \( \xi_{0.75} = 12.396 \).

3. Let \( X \) be the number of nonconforming units in the sample. Then assuming the process is in control, \( X \sim \text{Bi}(50, 0.02) \). The decision rule is: if one or more nonconforming units are found in a sample, the process is stopped and investigated for the presence of assignable causes. Now,

\[
P(X \geq 1 \mid p = 0.02) = 1 - P(X = 0) = 0.636.
\]

The assumption here is that a fraction nonconforming (\( p \) say) of 0.02 is the present inherent capability of the process. According to the decision rule, the process will be stopped 64\% of the time even when the process is operating in control. In other words, the technician will spend over half of his/her time on wild goose chases looking for assignable causes that don’t exist. **It’s a terrible decision rule.**

4. First find the probability that the decision rule in the previous problem will signal the technician to stop the process on any given sample when \( p = 0.04 \), i.e., when \( X \sim \text{Bi}(50, 0.04) \).

\[
P(X \geq 1 \mid p = 0.04) = 1 - P(X = 0) = 0.870.
\]

Let \( Y \) be the number of samples required for a signal to occur. Then \( Y \) has the geometric distribution with parameter 0.13 (see Sec. 2.2.4 and the geometric distribution as a special case of the Pascal). Then \( E(Y) = 1/0.87 = 1.15 \). Although on the average the change would be detected rather quickly, because of the result in Problem 3, it’s still a bad decision rule.

5.A. Let \( X = \) number of defects in a bottle. If \( X \sim \text{Po}(0.00001) \),

\[
P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.00001} = 0.00001.
\]

*Note: \( e^{-x} \approx 1 - x \) when \( x \) is small.*

B. Let \( Y = \) number of defective bottles in a shipment. Then, assuming that whether or not any bottle has a defect (at least one) is independent of whether any other has, \( Y \sim \text{bi}(500, 0.00001) \). Thus,
\[
P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{500}{0}(0.00001)^0(0.99999)^{500} = 0.00499.
\]

6.A. Let \( T \) = tensile strength. \( T \sim N(40, 64) \).

\[
P(T < 34) = \Phi \left( \frac{34 - 40}{8} \right) = \Phi(-0.75) = 0.2266.
\]

Let \( Y \) = number not meeting lower spec. Then, \( Y \sim \text{bi}(50000, 0.2266) \) and \( E(Y) = 50000(0.2266) = 11,330 \).

**Note:** We don’t seem to be dealing with very good companies, are we? Maybe they’ll get better as we proceed through the course.

B. \( P(T > 48) = \Phi \left( -\frac{48 - 40}{8} \right) = \Phi(-1) = 0.1587 \).

Thus, expected no. exceeding 48 = 50000(0.1587) = 7935.

C. Let \( T \sim N \left( 40, \sigma^2 \right) \). We need to find \( \sigma \) such that

\[
0.0001 = P(T < 34) = \Phi \left( \frac{34 - 40}{\sigma} \right) = \Phi \left( -\frac{6}{\sigma} \right)
\]

or, equivalently, \( \Phi \left( 6/\sigma \right) = 0.9999 \), which leads to \( 6/\sigma = 3.71 \) or \( \sigma = 1.62 \). (That’s 1.62 from 8!!! It ain’t gonna be easy.)

7. A. \( T_1 + T_2 + T_3 \sim N \left( 10 + 5 + 5, 1 + 0.2^2 + 0.3^2 \right) \equiv N(20, 1.13) \).

B. Let \( T \) = task completion time. Then by Part A,

\[
P(T \leq 22) = \Phi \left( \frac{22 - 20}{\sqrt{1.13}} \right) = \Phi(1.88) = 0.970.
\]

C. Let \( Y \) = no. not completed within the 22 minute standard. Then, \( Y \sim \text{bi}(20, 0.03) \). By the usual methods, e.g., use MINITAB, \( P(Y \geq 2) = 1 - P(Y \leq 1) = 0.120 \).
D. Let $T_i^*$ be John's new time. **Meaning 1:**

$$T_i^* = 0.8T_i \sim N(0.8 \times 10, 0.8^2 \times 1) \equiv N(0.8, 0.64).$$

Graphically,

**Meaning 2:** $T_i^* \sim N(8, 1)$, i.e., John shows a 20% improvement only on average! Quite a difference. *Say what you mean,* but perhaps when talking about random variables, some people don’t really know what they mean.

8.A. MINITAB output appears below. Since the p-value is less than 0.05, we would reject the hypothesis that the mean thickness is 34.0.

Test of $\mu = 13.4000$ vs $\mu \neq 13.4000$

<table>
<thead>
<tr>
<th>Variable</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th>SE Mean</th>
<th>SE Mean</th>
<th>T</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>THICKNESS</td>
<td>10</td>
<td>13.3962</td>
<td>0.0039</td>
<td>0.0012</td>
<td>-3.09</td>
<td>0.013</td>
<td></td>
</tr>
</tbody>
</table>

B. MINITAB output appears below.

<table>
<thead>
<tr>
<th>Variable</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th>SE Mean</th>
<th>99.0 % CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>THICKNESS</td>
<td>10</td>
<td>13.3962</td>
<td>0.0039</td>
<td>0.0012</td>
<td>(13.3922, 13.4002)</td>
</tr>
</tbody>
</table>
C. A probability plot and normality test (Anderson-Darling) appears below. Clearly, we would not reject the hypothesis of normality.

![Normal Probability Plot](image)

9. A and C. MINITAB output appears below. The p-value for the hypothesis test is very large. Consequently, we cannot reject the hypothesis that the (population) mean measurements are the same. The 95% confidence interval also appears in the output. Note that it includes 0 and is very narrow.

**Two-sample T for TECH1 vs TECH2**

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th>SE Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>TECH1</td>
<td>7</td>
<td>1.383</td>
<td>0.115</td>
<td>0.043</td>
</tr>
<tr>
<td>TECH2</td>
<td>8</td>
<td>1.376</td>
<td>0.125</td>
<td>0.044</td>
</tr>
</tbody>
</table>

\[
\text{Difference} = \mu_{\text{TECH1}} - \mu_{\text{TECH2}}
\]

Estimate for difference: 0.0066

95% CI for difference: (-0.1280, 0.1412)

T-Test of difference = 0 (vs not =): T-Value = 0.11  P-Value = 0.917  DF = 13

Both use Pooled StDev = 0.120

B. The practical implications are that we cannot reject the hypothesis that measurements made by the two technicians differ significantly. If we had rejected the null hypothesis, it would have meant that we have a serious problem in that the technicians’ measurements are not consistent. Perhaps one or both need training.
D. \( s_1 = 0.11485, s_2 = 0.12489. \)

\[
f_{0.975, 6, 7} = 0.1756 \leq f_0 = \left( \frac{0.11485}{0.12489} \right)^2 = 0.846 \leq f_{0.025, 6, 7} = 5.1186
\]

\[\therefore \text{Do not reject hypothesis of equality of variances.}\]

E. The 95\% confidence interval for \( \sigma^2_1 / \sigma^2_2 \) is

\[
\left[ \left( \frac{0.11485}{0.12489} \right)^2 f_{0.975, 7, 6}, \frac{0.11485}{0.12489} \right]^2 f_{0.025, 7, 6} \equiv \left[ \left( \frac{0.11485}{0.12489} \right)^2, \frac{0.11485}{0.12489} \right]^2 5.6955
\]

\[\equiv [0.1652, 4.817] \]

F. The 95\% confidence interval for \( \sigma^2_2 \) is

\[
\left[ \frac{7s^2_2}{\chi^2_{0.25/7}}, \frac{7s^2_2}{\chi^2_{0.75/7}} \right] \equiv \left[ \frac{7(0.12489)^2}{16.0128}, \frac{7(0.12489)^2}{1.6899} \right] \equiv [0.00681, 0.0646].
\]

G. Superposed probability plots appear below. The assumption of normality appears to be reasonable. Furthermore, the fitted lines are nearly identical indicating that the two technicians are, statistically speaking, making the same measurements.
10. **The intended problem was Exercise 3-19.** This is another important type of measurement system assessment so the solution follows. It is simply a paired $t$-test, paired because the *same* parts are being measured and consequently the measurements will be dependent. MINITAB output appears below. Since the p-value is (much) higher than 0.01, we cannot reject the hypothesis that the mean measurements are different at the 0.01 level of significance. Note how narrow the 99% confidence interval for the difference in means is and it does contain 0.

Paired T for MCAL - VCAL

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th>SE Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCAL</td>
<td>12</td>
<td>0.151167</td>
<td>0.000835</td>
<td>0.000241</td>
</tr>
<tr>
<td>VCAL</td>
<td>12</td>
<td>0.151583</td>
<td>0.001621</td>
<td>0.000468</td>
</tr>
<tr>
<td>Difference</td>
<td>12</td>
<td>-4.2E-04</td>
<td>0.001311</td>
<td>0.000379</td>
</tr>
</tbody>
</table>

99% CI for mean difference: (-0.001593, 0.000760)
T-Test of mean difference = 0 (vs not = 0): T-Value = -1.10  P-Value = 0.295