Quadratic Forms

A *quadratic form in* $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}'$ is a mathematical expression that can be written as $\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}x_i x_j$. For example, $3x_1^2 - 7x_1x_2 + 83x_2^2$ is a quadratic form, but $7 + 2x_2^2$ is not a quadratic form (because of the 7 term), and $3x_1^2 - 7x_1x_2 + 83x_2^2 + x_1^3$ is not a quadratic form (because of the x_1^3 term).

Quadratic forms can be rewritten in the form $\mathbf{x}'\mathbf{A}\mathbf{x}$ by choosing an appropriate $n \times n$ matrix $\mathbf{A}$. In particular, if you do the matrix multiplications, you will see that $\mathbf{x}'\begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix}\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}x_i x_j$. Check this yourself so you see that this works.

An important simplification can be made when using quadratic forms. Since $x_i x_j = x_j x_i$, relevant terms in the sum can be combined: $k_{ij}x_i x_j + k_{ji}x_j x_i = (k_{ij} + k_{ji})x_i x_j$, for each $i \neq j$. Therefore the whole quadratic form can be written as $\mathbf{x}'\begin{bmatrix} a_{11} & \frac{1}{2}a_{21} & \cdots & \frac{1}{2}a_{n1} \\ \frac{1}{2}a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{n1} & \frac{1}{2}a_{n2} & \cdots & \frac{1}{2}a_{nn} \end{bmatrix}\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij}x_i x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}x_i x_j$, where $a_{ij} = k_{ij} + k_{ji}$ if $i \neq j$ or $a_{ii} = k_{ii}$ if $i = j$. This simplification is important because now the matrix in the middle is symmetric, which often allows you to make simplifications when handling algebraic expressions involving the matrix. When we use quadratic forms, we will write the matrix in the middle as a symmetric matrix.