1. Consider the linear first-order PDE

\[(x + y)u_x - (y + 1)u_y = u - 2y, \quad y > 0\]

with boundary condition \(u(x, 0) = f(x)\).

(a) Determine characteristic curves of the PDE given by \(x = x(s)\) and \(y = y(s)\), where \(s\) is a parameter, such that \(x(0) = x_0\) and \(y(0) = 0\).

(b) Integrate the PDE along characteristics to determine the solution \(u(x, y)\) of the boundary-value problem.

**Solution**

(a) The characteristics follow from the following differential equations.

\[
\frac{dx}{ds} = x + y \quad ; \quad x(0) = x_0 \\
\frac{dy}{ds} = -y - 1 \quad ; \quad y(0) = 0
\]

The general solution of (2) is given by \(y(s) = Ce^{-s} - 1\) where \(C\) is a constant. Applying the condition \(y(0) = 0\) gives \(c = 1\). Therefore, \(y(s) = e^{-s} - 1\). Using this in (1), we obtain the following differential equation.

\[
\frac{dx}{ds} - x = e^{-s} - 1 \quad ; \quad x(0) = x_0
\]

The homogeneous solution of (2) is given by \(x_h(s) = Ce^s\) where \(C\) is some constant. The particular solution is given by \(x_p(s) = Ae^{-s} + B\) where \(A\) and \(B\) are constants to be determined. Using this form in (3) results \(A = -\frac{1}{2}\) and \(B = 1\). Therefore, the general solution of (3) is given by \(x(s) = Ce^s - \frac{1}{2}e^{-s} + 1\). Applying the condition \(x(0) = x_0\), it follows that \(C = x_0 - \frac{1}{2}\). Hence, \(x(s) = (x_0 - \frac{1}{2})e^s - \frac{1}{2}e^{-s} + 1\). In summary, the characteristic curves of the PDE given by

\[
x(s) = \left(x_0 - \frac{1}{2}\right)e^s - \frac{1}{2}e^{-s} + 1 \quad ; \quad y(s) = e^{-s} - 1
\]

(b) Along the characteristics, \(\frac{du}{ds} = u - 2y\). At \(s = 0\), \(x = x_0\) and \(y(0) = 0\). Therefore, \(u|_{s=0} = f(x_0)\). Using the solution for \(y(s)\) in (4), the differential equation that determines \(u(s)\) is given by

\[
\frac{du}{ds} - u = -2e^{-s} + 2 \quad ; \quad u(0) = f(x_0)
\]

The homogeneous solution of (5) is given by \(u_h(s) = Ce^s\) where \(C\) is some constant. The particular solution is given by \(u_p(s) = Ae^{-s} + B\) where \(A\) and \(B\) are constants to be determined. Using this form in (5) results \(A = 1\) and \(B = -2\). Therefore, the general solution of (5) is given by \(u(s) = Ce^s + e^{-s} - 2\). Applying the condition \(u(0) = f(x_0)\), it follows that \(C = f(x_0) + 1\). Hence, the solution \(u(s)\) is given by

\[
u(s) = (f(x_0) + 1)e^s + e^{-s} - 2
\]
To express the solution as \( u(x, y) \), note the following equations:

\[
e^{-s} = y + 1 \\
e^s = \frac{1}{y+1} \\
x = -\frac{1}{2}e^{-s} + \left( x_0 - \frac{1}{2} \right) e^s + 1 = -\frac{1}{2}(y+1) + \left( x_0 - \frac{1}{2} \right) \left( \frac{1}{y+1} \right) + 1 \\
x_0 = xe^{-s} + \frac{1}{2}e^{-2s} - e^{-s} + \frac{1}{2} = \frac{1}{2}y^2 + xy + x
\]

Using the above forms in (6) gives the final solution \( u(x, y) \) of the boundary-value problem

\[
\begin{bmatrix}
\begin{array}{c}
u(x, y) = \\
\end{array}
\end{bmatrix} = \
\begin{bmatrix}
f \left( \frac{1}{2}y^2 + xy + x \right) + 1 \\
\left( \frac{1}{y+1} \right) + y - 1
\end{bmatrix}
\]

2. A tiny bit of linear algebra for your good health (and because it is relevant for systems of first-order PDEs). Consider the following \( 2 \times 2 \) matrices.

\[
A = \begin{bmatrix}
3 & -2 \\
4 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -1 \\
1 & 3
\end{bmatrix}, \quad C = \begin{bmatrix}
-1 & 9 \\
3 & 5
\end{bmatrix}
\]

Find the eigenvalues for each matrix. If the eigenvalues are real, then determine whether the matrix is diagonalizable.

**Solution**

(a) The characteristic equation for \( A \) is given by \( \lambda^2 - 2\lambda + 5 = 0 \). It follows that \( \lambda_1 = 1 + 2i \) and \( \lambda_2 = 1 - 2i \).

(b) The characteristic equation for \( B \) is given by \( \lambda^2 - 4\lambda + 4 = 0 \). It follows that \( \lambda_1 = \lambda_2 = 2 \).

It can be checked that \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) is the only eigenvector of this matrix. Therefore, \( B \) is not diagonalizable.

(c) The characteristic equation for \( C \) is given by \( \lambda^2 - 4\lambda - 32 = 0 \). It follows that \( \lambda_1 = -4 \) and \( \lambda_2 = 8 \). Since \( C \) has distinct real eigenvalues, the matrix is diagonalizable.

**Remark:** If a matrix has real distinct eigenvalues, it is diagonalizable. If a matrix has real repeated eigenvalues, it may or may not be diagonalizable. For example, the matrix \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) has 1 as a repeated eigenvalue but it is trivially diagonalizable. On the other hand, the matrix \( B \) considered above has repeated eigenvalues but is not diagonalizable. For the case of repeated roots, diagonalizability is determined by showing that the eigenvectors are independent.

3. Consider the system of first-order PDEs

\[
\begin{bmatrix}
3 & -1 \\
4 & -2
\end{bmatrix} u_x = 0, \quad t > 0
\]
with initial conditions
\[ u(x, 0) = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} \]

(a) Find a nonsingular matrix \( R \) such that the PDE for the variable \( w(x, t) = R^{-1}u(x, t) \) is diagonal. Find the general solution for \( w(x, t) \) using the method of characteristics.

(b) Find the solution for \( u(x, t) = Rw(x, t) \) using the initial conditions.

**Solution**

(a) Consider the coefficient matrix \( A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \). The characteristic equation for \( A \) is given by \( \lambda^2 - \lambda - 2 = 0 \). It follows that \( \lambda_1 = -1 \) and \( \lambda_2 = 2 \). The eigenvectors of \( A \) are computed as follows.

\[
\begin{align*}
(A - \lambda_1 I)r_1 &= 0 \\
\begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix} r_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
r_1 &= \begin{bmatrix} 1 \\ 4 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
(A - \lambda_2 I)r_2 &= 0 \\
\begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} r_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
r_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{align*}
\]

The nonsingular matrix \( R \) is made up of the eigenvectors and is given by

\[
R = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}
\]

Consider the system \( u_t + Au_x = 0 \). The inverse of the matrix \( R \) is given by

\[
R^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}
\]

Multiplying both sides of the system by \( R^{-1} \) results \( R^{-1}u_t + R^{-1}Au_x = 0 \). Equivalently, the system can be written as \( R^{-1}u_t + (R^{-1}AR)R^{-1}u_x = 0 \). Defining the variable \( w = R^{-1}u \) and noting that \( R^{-1}AR = \Lambda \) where \( \Lambda \) is the eigenvalue matrix, we have: \( \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} w_x = 0 \). The characteristics are given by

\[
\begin{align*}
&1. \quad \frac{dx}{dt} = \lambda_1 = -1 \rightarrow x = -t + \text{const} \\
&2. \quad \frac{dx}{dt} = \lambda_2 = 2 \rightarrow x = 2t + \text{const}
\end{align*}
\]

At \( t = 0 \), \( w(x, 0) = R^{-1}u(x, 0) = \frac{1}{3} \begin{bmatrix} g(x) - f(x) \\ 4f(x) - g(x) \end{bmatrix} \). In summary, to solve for \( w(x, t) \), the following system is considered.

\[
\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} w_x = 0 \quad ; \quad w(x, 0) = \frac{1}{3} \begin{bmatrix} g(x) - f(x) \\ 4f(x) - g(x) \end{bmatrix}
\]

Using the characteristics and the initial condition, the solution has the following form.

\[
\begin{align*}
w_1(x, t) &= w_1(x + t, 0) = -\frac{1}{3}f(x + t) + \frac{1}{3}g(x + t) \\
w_2(x, t) &= w_2(x - 2t, 0) = \frac{4}{3}f(x - 2t) - \frac{1}{3}g(x - 2t)
\end{align*}
\]
(b) The solution \( \mathbf{u}(x,t) \) is given by

\[
\mathbf{u}(x,t) = R \mathbf{w}(x,t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} f(x + t) + \frac{1}{3} g(x + t) \\ \frac{4}{3} f(x - 2t) - \frac{4}{3} g(x - 2t) \end{bmatrix} = -\frac{1}{3} f(x + t) + \frac{4}{3} f(x - 2t) + \frac{1}{3} g(x + t) - \frac{1}{3} g(x - 2t)
\]

4. Consider the system of first-order PDEs

\[
\mathbf{u}_t + A \mathbf{u}_x = 0, \quad A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, \quad x > 0, \quad t > 0
\]

(a) Verify that the characteristic velocities (i.e. the eigenvalues of \( A \)) are both real and positive.

(b) Since the characteristic velocities are both positive, we require conditions on both components of \( \mathbf{u}(x,t) \) along the initial line, \( t = 0 \) and \( x > 0 \), and along the boundary line, \( x = 0 \) and \( t > 0 \). Assume that

\[
\mathbf{u}(x,0) = \begin{bmatrix} f(x) \\ 0 \end{bmatrix}, \quad \mathbf{u}(0,t) = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}
\]

Diagonalize the system and use the method of characteristics to find the solution for \( \mathbf{u}(x,t) \).

Solution

(a) The characteristic equation for \( A \) is given by \( \lambda^2 - 6\lambda + 8 = 0 \). It follows that \( \lambda_1 = 2 \) and \( \lambda_2 = 4 \). Therefore, the characteristics velocities (i.e. eigenvalues of \( A \)) are both real and positive.

(b) The eigenvectors of \( A \) are computed as follows.

\[
(A - \lambda_1 I) \mathbf{r}_1 = \mathbf{0} \quad \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \mathbf{r}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{r}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]

\[
(A - \lambda_2 I) \mathbf{r}_2 = \mathbf{0} \quad \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{r}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

The nonsingular matrix \( R \) is made up of the eigenvectors and is given by

\[
R = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}
\]

Consider the system \( \mathbf{u}_t + A \mathbf{u}_x = \mathbf{0} \). The inverse of the matrix \( R \) is given by

\[
R^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -3 & -1 \end{bmatrix}
\]

Multiplying both sides of the system by \( R^{-1} \) results \( R^{-1} \mathbf{u}_t + R^{-1} A \mathbf{u}_x = \mathbf{0} \). Equivalently, the system can be written as \( R^{-1} \mathbf{u}_t + (R^{-1} A) R^{-1} \mathbf{u}_x = \mathbf{0} \). Defining the variable \( \mathbf{w} = \)
\[ R^{-1}u \] and noting that \( R^{-1}AR = \Lambda \) where \( \Lambda \) is the eigenvalue matrix, we have: \( \dot{w} + \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} w = 0 \). The characteristics are given by

1. \( \frac{dx}{dt} = \lambda_1 = 2 \rightarrow x = 2t + \text{const} \)

2. \( \frac{dx}{dt} = \lambda_2 = 4 \rightarrow x = 4t + \text{const} \)

At \( t = 0 \), \( w(x, 0) = R^{-1}u(x, 0) = \begin{bmatrix} -\frac{1}{2}f(x) \\ \frac{3}{2}f(x) \end{bmatrix} \). At \( x = 0 \), \( w(0, t) = R^{-1}u(0, t) = \begin{bmatrix} \frac{1}{2}g(t) \\ -\frac{1}{2}g(t) \end{bmatrix} \).

In summary, to solve for \( w(x,t) \), the following system is considered.

\[ \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} w = 0 \quad ; \quad w(x,0) = \begin{bmatrix} -\frac{1}{2}f(x) \\ \frac{3}{2}f(x) \end{bmatrix} \quad ; \quad w(0,t) = \begin{bmatrix} \frac{1}{2}g(t) \\ -\frac{1}{2}g(t) \end{bmatrix} \]

The solution is determined in cases.

**x > 4t:** In this region, the solution for \( w \) is fully determined from the initial conditions.

Using the characteristics and the initial condition, the solution has the following form.

\[
w_1(x,t) = w_1(x-2t,0) = -\frac{1}{2}f(x-2t) \\
w_2(x,t) = w_2(x-4t,0) = \frac{3}{2}f(x-4t)
\]

The solution \( u(x,t) \) is given by

\[
u(x,t) = Rw(x,t) = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}f(x-2t) \\ \frac{3}{2}f(x-4t) \end{bmatrix} \\
= \begin{bmatrix} -\frac{1}{2}f(x-2t) + \frac{3}{2}f(x-4t) \\ -\frac{3}{2}f(x-2t) + \frac{3}{2}f(x-4t) \end{bmatrix}
\]

**2t < x < 4t:** In this region, \( w_1 \) is determined from the initial condition and \( w_2 \) is determined from the boundary condition. Using the characteristics, the initial condition and the boundary condition, the solution has the following form.

\[
w_1(x,t) = w_1(x-2t,0) = -\frac{1}{2}f(x-2t) \\
w_2(x,t) = w_2\left(0,t - \frac{x}{4}\right) = -\frac{1}{2}g\left(t - \frac{x}{4}\right)
\]

The solution \( u(x,t) \) is given by

\[
u(x,t) = Rw(x,t) = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}f(x-2t) \\ -\frac{1}{2}g\left(t - \frac{x}{4}\right) \end{bmatrix} \\
= \begin{bmatrix} -\frac{1}{2}f(x-2t) - \frac{1}{2}g\left(t - \frac{x}{4}\right) \\ -\frac{3}{2}f(x-2t) - \frac{1}{2}g\left(t - \frac{x}{4}\right) \end{bmatrix}
\]
In this region, the solution for $w$ is fully determined from the boundary conditions. Using the characteristics and the boundary condition, the solution has the following form.

\[
w_1(x, t) = w_1(0, t - \frac{x}{2}) = \frac{1}{2} g(t - \frac{x}{2})
\]
\[
w_2(x, t) = w_2(0, t - \frac{x}{4}) = -\frac{1}{2} g(t - \frac{x}{4})
\]

The solution $u(x, t)$ is given by

\[
u(x, t) = R w(x, t) = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} g(t - \frac{x}{2}) \\ -\frac{1}{2} g(t - \frac{x}{4}) \end{bmatrix}
\]
\[
= \begin{bmatrix} \frac{1}{2} g(t - \frac{x}{2}) - \frac{1}{2} g(t - \frac{x}{4}) \\ \frac{3}{2} g(t - \frac{x}{2}) - \frac{1}{2} g(t - \frac{x}{4}) \end{bmatrix}
\]