1. Use a suitable eigenfunction expansion to determine the temperature \( u(r,t) \) satisfying

\[
\frac{\partial u}{\partial t} = k \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + Q(r,t), \quad a < r < b, \quad t > 0
\]

with initial condition \( u(r,0) = f(r) \) and boundary conditions \( u(a,t) = u(b,t) = 0 \). Is it valid to differentiate the eigenfunction expansion term-by-term?

**Solution** Consider the solution of the homogeneous PDE \( \frac{\partial u}{\partial t} = k \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \) with the given boundary conditions. Using the method of separation of variables, assume \( u(r,t) = R(r)T(t) \).

Substituting this product form in the homogeneous PDE and simplifying, we obtain

\[
\frac{T'}{kT} = \frac{(rR')'}{Rr} = -\lambda
\]

where \( \lambda \) is some constant. (1) results the following two ordinary differential equations. The first ODE resulting from (1) is for \( R(r) \) and is given by \( rR'' + rR' + \lambda r^2 R = 0 \). Equivalently, we have \( r^2R'' + rR' + \lambda r^2 R = 0 \). We now consider the boundary conditions. The first boundary condition \( u(a,t) = 0 \) implies that \( R(a)T(t) = 0 \). For nontrivial solutions, set \( R(a) = 0 \). In an analogous manner, the second boundary condition \( u(b,t) = 0 \) implies that \( R(b) = 0 \). With this, consider the following boundary value problem.

\[
r^2R'' + rR' + \lambda r^2 R = 0 \quad (2)
\]

\[
R(a) = 0 \quad (3)
\]

\[
R(b) = 0 \quad (4)
\]

The general solution of (2) is given by \( R(r) = c_1 J_0(\sqrt{\lambda} r) + c_2 Y_0(\sqrt{\lambda} r) \) where \( c_1 \) and \( c_2 \) are some constants. Applying the first boundary condition in (3), \( R(a) = c_1 J_0(\sqrt{\lambda} a) + c_2 Y_0(\sqrt{\lambda} a) = 0 \). Similarly, applying the second boundary condition in (4), \( R(b) = c_1 J_0(\sqrt{\lambda} b) + c_2 Y_0(\sqrt{\lambda} b) = 0 \). A minor algebra results

\[
c_1 \left( J_0(\sqrt{\lambda} b) - \frac{J_0(\sqrt{\lambda} a)}{Y_0(\sqrt{\lambda} a)} Y_0(\sqrt{\lambda} b) \right) = 0 \quad (5)
\]

Let \( \lambda_n \) denote the eigenvalue corresponding to the \( n \)-th zero of the above equation. The eigenfunctions are given by

\[
R_n(r) = J_0(\sqrt{\lambda_n} r) - \frac{J_0(\sqrt{\lambda_n} a)}{Y_0(\sqrt{\lambda_n} a)} Y_0(\sqrt{\lambda_n} r) \quad (6)
\]

Having determined \( \lambda_n \) and \( R_n(r) \), we now consider the non-homogeneous PDE. The starting point is the following assumption on the form of \( u(r,t) \) and \( Q(r,t) \).

\[
u(r,t) = \sum_{n=1}^{\infty} c_n(t)R_n(r) \quad ; \quad Q(r,t) = \sum_{n=1}^{\infty} q_n(t)R_n(r) \quad (7)
\]

where \( c_n(t) \) is a time dependent coefficient to be determined. The coefficient \( q_n(t) \) is given by

\[
q_n(t) = \frac{\int_a^b Q(r,t)R_n(r)rdr}{\int_a^b R_n(r)^2rdr} \quad (8)
\]
Substituting the forms in (7) in the PDE, we obtain
\[ \sum_{n=1}^{\infty} c_n'(t)R_n(r) = \frac{k}{r} \sum_{n=1}^{\infty} c_n(t)(rR_n')' + \sum_{n=1}^{\infty} q_n(t)R_n(r) = \sum_{n=1}^{\infty} -c_n(t)\lambda_n kR_n + \sum_{n=1}^{\infty} q_n(t)R_n(r) \]
where we have used the fact that \((rR_n')' = -\lambda_n rR_n\). The above equation can also be equivalently written as follows
\[ \sum_{n=1}^{\infty} [c_n' + k\lambda_n c_n - q_n]R_n(r) = 0 \]
The above equation holds only if \(c_n' + k\lambda_n c_n - q_n = 0\). In addition, using the form of \(u(r, t)\) in (7), note that \(u(r, 0) = \sum_{n=1}^{\infty} c_n(0)R_n(r) = f(r)\). It follows that \(c_n(0) = \frac{\int_a^b f(r)R_n(r)rdr}{\int_a^b R_n(r)^2rdr}\). With this, consider the following initial value problem.
\[ c_n' + k\lambda_n c_n - q_n = 0 \quad ; \quad c_n(0) = \frac{\int_a^b f(r)R_n(r)rdr}{\int_a^b R_n(r)^2rdr} \quad (9) \]
We use the method of integrating factors to solve (9) obtaining the following solution.
\[ c_n(t) = e^{-k\lambda_n t} \int_0^t e^{k\lambda_n \tau} q_n(\tau)d\tau + Ke^{-k\lambda_n t} \]
Applying the initial condition on the last equation, we have \(K = c_n(0)\). Therefore, \(c_n\) is given by
\[ c_n(t) = c_n(0)e^{-k\lambda_n t} + \int_0^t e^{-k\lambda_n (t-\tau)} q_n(\tau)d\tau \]
Having determined \(c_n\), the solution \(u(x, t)\) of the non-homogeneous PDE can be represented as follows
\[ u(r, t) = \sum_{n=1}^{\infty} c_n(0)e^{-k\lambda_n t} + \int_0^t e^{-k\lambda_n (t-\tau)} q_n(\tau)d\tau R_n(r) \quad (10) \]
with \(q_n\) given in (8) and \(c_n(0) = \frac{\int_a^b f(r)R_n(r)rdr}{\int_a^b R_n(r)^2rdr}\).

**Remark:** In the above calculations, the eigenfunction expansion can be differentiated term by term since \(u(r, t)\) solves the same homogeneous boundary conditions as does \(R_n(r)\).
2. Text exercises 8.5.2 and 8.5.3, pages 364–365.

Solution

(a) Consider the solution of the homogeneous PDE \( \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \) with the given boundary conditions. Using the method of separation of variables, assume \( u(x, t) = \phi(x)T(t) \). Substituting this product form in the homogeneous PDE and simplifying, we obtain

\[
\frac{T''}{c^2 T} = \frac{\phi''}{\phi} = -\lambda
\]

where \( \lambda \) is some constant. (11) results the following two ordinary differential equations.

The first ODE resulting from (11) is for \( \phi(r) \) and is given by

\[
\phi'' + \lambda \phi = 0
\]

We now consider the boundary conditions. The first boundary condition \( u(0, t) = 0 \) implies that \( \phi(0)T(t) = 0 \). For nontrivial solutions, set \( \phi(0) = 0 \). In an analogous manner, the second boundary condition \( u(L, t) = 0 \) implies that \( \phi(L) = 0 \). With this, consider the following boundary value problem.

\[
\phi'' + \lambda \phi = 0 \quad (12) \\
\phi(0) = 0 \quad (13) \\
\phi(L) = 0 \quad (14)
\]

The above boundary problem can be readily solved. The eigenvalues are given by

\[
\lambda_n(x) = \left( \frac{n\pi}{L} \right)^2
\]

with corresponding eigenfunctions

\[
\phi_n(x) = \sin \frac{n\pi x}{L}
\]

Having determined \( \lambda_n \) and \( \phi_n(x) \), we now consider the non-homogeneous PDE. The starting point is the following assumption on the form of \( u(x, t) \) and \( Q(x, t) \).

\[
u(x, t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x) \quad ; \quad Q(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)
\]

where \( c_n(t) \) is a time dependent coefficient to be determined. The coefficient \( q_n(t) \) is given by

\[
q_n(t) = \frac{2}{L} \int_0^L Q(x, t) \sin \frac{n\pi x}{L} \, dx
\]

Substituting the forms in (17) in the PDE, we obtain

\[
\sum_{n=1}^{\infty} c_n''(t)\phi_n(x) = c^2 \sum_{n=1}^{\infty} c_n(t)\phi''(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x) = \sum_{n=1}^{\infty} -c_n(t)\lambda_n c^2 \phi_n(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x)
\]

where we have used the fact that \( \phi''(x) = -\lambda_n \phi(x) \). The above equation can also be equivalently written as follows

\[
\sum_{n=1}^{\infty} \left[ c_n'' + \lambda_n c^2 c_n - q_n \right] \phi_n(x) = 0
\]
The above equation holds only if $c''_n + \lambda_n c_n = 0$. In addition, using the form of $u(x, t)$ in (17), note that $u(x, 0) = \sum_{n=1}^{\infty} c_n(0) \phi_n(x) = f(x)$. It follows that $c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) dx$. The condition $u_t(x, 0) = 0$ implies that $c'_n(0) = 0$. With this, consider the following initial value problem.

$$c''_n + \lambda_n c_n = q_n ; \quad c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) dx ; \quad c'_n(0) = 0$$  \hspace{1cm} (19)

To solve the above problem, we use variation of parameters. The homogeneous solutions are given by $c_1 = \cos \sqrt{\lambda_n}ct$ and $c_2 = \sin \sqrt{\lambda_n}ct$. Using variation of parameters, the particular solution is given by

$$c'_n(t) = -c_1 \int \frac{c_2 q_n}{W(c_1, c_2)} dt + c_2 \int \frac{c_1 q_n}{W(c_1, c_2)} dt$$  \hspace{1cm} (20)

First note that

$$W(c_1, c_2) = \det \begin{bmatrix} c_1 & c_2 \\ c'_1 & c'_2 \end{bmatrix} = \det \begin{bmatrix} \cos \sqrt{\lambda_n}ct & \sin \sqrt{\lambda_n}ct \\ -\sqrt{\lambda_n}c \sin \sqrt{\lambda_n}ct & \sqrt{\lambda_n}c \cos \sqrt{\lambda_n}ct \end{bmatrix} = \sqrt{\lambda_n}c$$

Using the variation of parameters formula in (20), we obtain the following particular solution

$$c'_n(t) = -\frac{\cos \sqrt{\lambda_n}ct}{a_n} \int_0^t \sin \sqrt{\lambda_n}c \tau q_n(\tau) d\tau + \frac{\sin \sqrt{\lambda_n}ct}{a_n} \int_0^t \cos \sqrt{\lambda_n}c \tau q_n(\tau) d\tau$$

$$= \int_0^t q_n(\tau) \sin \sqrt{\lambda_n}c(t - \tau) \frac{1}{c \sqrt{\lambda_n}}$$

with the latter equation following from applying the sine addition formula. With this, the general solution for (19) can be written as follows

$$c_n(t) = A \cos \sqrt{\lambda_n}ct + B \sin \sqrt{\lambda_n}ct + \int_0^t q_n(\tau) \sin \sqrt{\lambda_n}c(t - \tau) \frac{1}{c \sqrt{\lambda_n}}$$  \hspace{1cm} (21)

where $A$ and $B$ are constants to be determined. Applying the condition $c'_n(0) = 0$, we have $c'_n(0) = B = 0$. Applying the condition $c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) dx$, we have $c_n(0) = A$. Having determined the constants $A$ and $B$, the solution in (21) can be written as follows

$$c_n(t) = c_n(0) \cos \sqrt{\lambda_n}ct + \int_0^t q_n(\tau) \sin \sqrt{\lambda_n}c(t - \tau) \frac{1}{c \sqrt{\lambda_n}}$$

Finally, the solution $u(x, t)$ of the non-homogeneous PDE can be represented as follows

$$u(x, t) = \sum_{n=1}^{\infty} c_n(0) \cos \sqrt{\lambda_n}ct + \int_0^t q_n(\tau) \sin \sqrt{\lambda_n}c(t - \tau) \frac{1}{c \sqrt{\lambda_n}} \sin \frac{n\pi x}{L}$$

with $q_n$ given in (18) and $c_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$. 

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(b) Once again, the starting point is the following assumption on the form of \( u(x,t) \) and \( Q(x,t) \).

\[
u(x,t) = \sum_{n=1}^{\infty} c_n(t) \phi_n(x) \quad ; \quad Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \phi_n(x) \tag{22}\]

where \( c_n(t) \) is a time dependent coefficient to be determined. The coefficient \( q_n(t) \) is given by

\[
q_n(t) = \frac{2}{L} \int_0^L Q(x,t) \sin \frac{n \pi x}{L} \, dx = \cos \omega t \left\{ 2 \int_0^L g(x) \sin \frac{n \pi x}{L} \, dx \right\}
\]

Following the same analysis as (a), the following initial value problem is considered to solve for \( u(x,t) \).

\[
c_n'' + \lambda_n c_n = g_n \cos \omega t \quad ; \quad c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx \quad ; \quad c_n'(0) = 0 \tag{24}\]

The homogeneous solutions are given by \( A_n \cos \sqrt{\lambda_n} ct + B_n \sin \sqrt{\lambda_n} ct \) where \( A_n \) and \( B_n \) are constants to be determined. To find the particular solution, using the method of undetermined coefficients, set \( c_n'(t) = K_n \cos \omega t \) where \( K_n \) is some constant. It can be easily verified that \( K_n = \frac{g_n}{\lambda_n c^2 - w^2} \). With this, the general solution for (24) can be written as follows

\[
c_n(t) = A_n \cos \sqrt{\lambda_n} ct + B_n \sin \sqrt{\lambda_n} ct + \frac{g_n}{\lambda_n c^2 - w^2} \cos \omega t \tag{25}\]

Applying the condition \( c_n'(0) = 0 \), we have \( c_n'(0) = B_n = 0 \). Applying the condition \( c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx \), we have \( c_n(0) = A_n + \frac{g_n}{\lambda_n c^2 - w^2} \). Having determined the constants \( A \) and \( B \), the solution in (25) can be written as follows

\[
c_n(t) = \left( c_n(0) - \frac{g_n}{\lambda_n c^2 - w^2} \right) \cos \sqrt{\lambda_n} ct + \frac{g_n}{\lambda_n c^2 - w^2} \cos \omega t
\]

Finally, the solution \( u(x,t) \) of the non-homogeneous PDE can be represented as follows

\[
u(x,t) = \sum_{n=1}^{\infty} \left[ \left( c_n(0) - \frac{g_n}{\lambda_n c^2 - w^2} \right) \cos \sqrt{\lambda_n} ct + \frac{g_n}{\lambda_n c^2 - w^2} \cos \omega t \right] \sin \frac{n \pi x}{L}
\]

with \( q_n \) given in (23) and \( c_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx \).

**Resonance** Resonance occurs when \( \omega^2 = \lambda_n c^2 \). With this, we revisit the initial value problem considered in (24)

\[
c_n'' + w^2 c_n = g_n \cos \omega t \quad ; \quad c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx \quad ; \quad c_n'(0) = 0 \tag{26}\]

As before, the homogeneous solutions are given by \( A \cos \sqrt{\lambda_n} ct + B \sin \sqrt{\lambda_n} ct \) where \( A \) and \( B \) are constants to be determined. To find the particular solution, using the method of undetermined coefficients, set \( c_n'(t) = K t \sin \omega t \) where \( K \) is some constant. It can be
easily verified that $K = \frac{gn}{2w}$. With this, the general solution for (26) can be written as follows

$$c_n(t) = A \cos \sqrt{\lambda_n} ct + B \sin \sqrt{\lambda_n} ct + \frac{gn}{2w} t \sin wt$$

(27)

Applying the condition $c_n'(0) = 0$, we have $c_n'(0) = B = 0$. Applying the condition $c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) dx$, we have $c_n(0) = A$. Having determined the constants $A$ and $B$, the solution in (27) can be written as follows

$$c_n(t) = c_n(0) \cos \sqrt{\lambda_n} ct + \frac{gn}{2w} t \sin wt$$

Finally, the solution $u(x,t)$ of the non-homogeneous PDE can be represented as follows

$$u(x,t) = \sum_{n=1}^{\infty} \left[ c_n(0) \cos \sqrt{\lambda_n} ct + \frac{gn}{2w} t \sin wt \right] \sin \frac{n\pi x}{L}$$

with $q_n$ given in (23) and $c_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$.

8.5.3(a) Consider the solution of the homogeneous PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

with the given boundary conditions. Following the same analysis as 2(a), noting that the damping term only alters the time dependent solution, the eigenvalues are given by

$$\lambda_n(x) = \left(\frac{n\pi}{L}\right)^2$$

(28)

with corresponding eigenfunctions

$$\phi_n(x) = \sin \frac{n\pi x}{L}$$

(29)

Having determined $\lambda_n$ and $\phi_n(x)$, we now consider the non-homogeneous PDE. The starting point is the following assumption on the form of $u(x,t)$ and the non-homogeneous term denoted by $Q(x,t)$.

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \phi_n(x) ; \quad Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \phi_n(x)$$

(30)

where $c_n(t)$ is a time dependent coefficient to be determined. The coefficient $q_n(t)$ is given by

$$q_n(t) = \frac{2}{L} \int_0^L Q(x,t) \sin \frac{n\pi x}{L} dx = \cos wt \left\{ \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx + g_n \right\}$$

(31)

Substituting the forms in (30) in the PDE, we obtain

$$\sum_{n=1}^{\infty} c''_n(t) \phi_n(x) = c^2 \sum_{n=1}^{\infty} c_n(t) \phi''_n(x) - \beta \sum_{n=1}^{\infty} c'_n(t) \phi_n(x) + \sum_{n=1}^{\infty} q_n \phi_n(x)$$

where we have used the fact that $\phi''_n(x) = -\lambda_n \phi(x)$. The above equation can also be equivalently written as follows

$$\sum_{n=1}^{\infty} \left[ c''_n + \beta c'_n + \lambda_n c^2 c_n - q_n \right] \phi_n(x) = 0$$
The above equation holds only if \( c_n'' + \beta c_n' + \lambda_n c_n = 0 \). In addition, using the form of \( u(x, t) \) in (30), note that \( u(x, 0) = \sum_{n=1}^{\infty} c_n(0) \phi_n(x) = f(x) \). It follows that \( c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx \). The condition \( u_t(x, 0) = 0 \) implies that \( c_n'(0) = 0 \). With this, consider the following initial value problem.

\[
c_n'' + \beta c_n' + \lambda_n c_n = g_n \cos wt \quad ; \quad c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx \quad ; \quad c_n'(0) = 0 \quad (32)
\]

To find the homogeneous solutions, consider the characteristic equation \( r^2 + \beta r + \lambda_n c_n = 0 \) which gives the following roots

\[
r_1 = \frac{-\beta + \sqrt{k_n}}{2} \quad ; \quad r_2 = \frac{-\beta - \sqrt{k_n}}{2} \quad ; \quad k_n = \beta^2 - 4\lambda_n c_n
\]

Using the assumption \( 0 < \beta < \frac{2\pi}{L} \), \( k_n < 0 \) for all \( n \). Therefore, the roots \( r_1 \) and \( r_2 \) are complex. With this, the homogeneous solution is given by \( A_n e^{-\beta t_2} \cos k_n t + B_n e^{-\beta t_2} \sin k_n t \) where \( A_n \) and \( B_n \) are constants to be determined. To find the particular solution, using the method of undetermined coefficients, set \( c_n'(t) = E_n \cos wt + F_n \sin wt \) where \( E_n \) and \( F_n \) are some constants. After some algebra, it follows that

\[
E_n = \frac{(\lambda_n c_n^2 - w^2) g_n}{(\lambda_n c_n^2 - w^2)^2 + \beta^2 w^2} \quad ; \quad F_n = \frac{\beta w g_n}{(\lambda_n c_n^2 - w^2)^2 + \beta^2 w^2} \quad (33)
\]

The general solution for (32) can be written as follows

\[
c_n(t) = A_n e^{-\beta t_2} \cos k_n t + B_n e^{-\beta t_2} \sin k_n t + E_n \cos wt + F_n \sin wt \quad (34)
\]

Applying the condition \( c_n'(0) = 0 \), we have \( c_n'(0) = -\frac{\beta}{2} A_n + k_n B + F_n = 0 \). Applying the condition \( c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx \), we have \( c_n(0) = A + E_n \). It follows that

\[
A_n = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx - E_n \quad ; \quad B_n = \frac{\beta}{2k_n} \left[ \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx - E_n \right] - \frac{w}{k_n} F_n \quad (35)
\]

with \( E_n \) and \( F_n \) given in (33). Having determined all the constants, the solution in (25) can be written as follows

\[
c_n(t) = A_n e^{-\beta t_2} \cos k_n t + B_n e^{-\beta t_2} \sin k_n t + E_n \cos wt + F_n \sin wt
\]

Finally, the solution \( u(x, t) \) of the non-homogeneous PDE can be represented as follows

\[
u(x, t) = \sum_{n=1}^{\infty} \left[ A_n e^{-\beta t_2} \cos k_n t + B_n e^{-\beta t_2} \sin k_n t + E_n \cos wt + F_n \sin wt \right] \sin \frac{n\pi x}{L}
\]

with \( q_n \) given in (31), \( c_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \) and the constants \( E_n, F_n, A_n \) and \( B_n \) are given in (33) and (35).
8.5.3(b) As time gets large, the solution in 8.5.3(a) decays due to the exponential damping term. When $\beta \to 0$, the obtained solution is exactly the same as the non-resonant solution in 8.5.2(b). There is no resonance as $(\lambda_n c^2 - w^2)^2 + \beta^2 w^2 = 0$ does not have a solution for $w \in \mathcal{R}$. The physical interpretation is that unbounded growth of the solution is limited by dampening.

3. Text exercise 8.5.4, part (c) only, page 365.

**Solution** Consider the solution of the related homogeneous PDE $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with the given boundary conditions. Using the method of separation of variables, assume $u(x, t) = \phi(x) T(t)$. Substituting this product form in the homogeneous PDE and simplifying, we obtain

$$\frac{T''}{c^2 T} = \frac{\phi''}{\phi} = -\lambda$$

where $\lambda$ is some constant. (36) results the following two ordinary differential equations. The first ODE resulting from (36) is for $\phi(r)$ and is given by $\phi'' + \lambda \phi = 0$. We now consider the boundary conditions. The first related homogeneous boundary condition $u_x(0, t) = 0$ implies that $\phi'(0) T(t) = 0$. For nontrivial solutions, set $\phi'(0) = 0$. In an analogous manner, the second boundary condition $u(L, t) = 0$ implies that $\phi(L) = 0$. With this, consider the following boundary value problem.

$$\phi'' + \lambda \phi = 0$$
$$\phi'(0) = 0$$
$$\phi(L) = 0$$

The above boundary problem can be readily solved. The eigenvalues are given by

$$\lambda_n(x) = \left[\frac{(2n - 1)\pi}{2L}\right]^2 \quad n = 1, 2, 3, \ldots$$

with corresponding eigenfunctions

$$\phi_n(x) = \cos \frac{(2n - 1)\pi x}{2L}$$

Having determined $\lambda_n$ and $\phi_n(x)$, we now consider the non-homogeneous PDE. The starting point is the following assumption on the form of $u(x, t)$ and $Q(x, t)$.

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) \phi_n(x) \quad ; \quad c_n(x, t) = \sum_{n=1}^{\infty} q_n(t) \phi_n(x)$$

where $c_n(t)$ and $q_n(t)$ are given by

$$q_n(t) = \frac{2}{L} \int_{0}^{L} Q(x, t) \cos \frac{(2n - 1)\pi x}{2L} \, dx \quad ; \quad c_n(t) = \frac{2}{L} \int_{0}^{L} u(x, t) \cos \frac{(2n - 1)\pi x}{2L} \, dx$$

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The next step is to multiply the non-homogeneous PDE by \( \frac{2}{L} \phi_n(x) \) and integrate from \( x = 0 \) to \( x = L \).

\[
\frac{2}{L} \int_0^L u_{tt} \phi_n(x) \, dx = \frac{2c^2}{L} \int_0^L u_{xx} \phi_n(x) \, dx + q_n(t)
\]

\[
c''_n = \frac{2c^2}{L} u_x \phi_n(x) \bigg|_0^L - \frac{2c^2}{L} \int_0^L \phi_n'(x) u_x \, dx + q_n(t)
\]

\[
c''_n = -\frac{2c^2}{L} A(t) - \frac{2c^2}{L} \phi_n'(x) u|_0^L + \frac{2c^2}{L} \int_0^L \phi''_n(x) u(x) \, dx + q_n(t)
\]

\[
c''_n = -\frac{2c^2}{L} A(t) - c^2 \lambda_n c_n + q_n(t)
\]

In addition, using the form of \( u(x,t) \) in (42), note that \( u(x,0) = \sum_{n=1}^{\infty} c_n(0) \phi_n(x) = f(x) \). It follows that \( c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx \). The condition \( u_t(x,0) = 0 \) implies that \( c'_n(0) = 0 \). With this, consider the following initial value problem.

\[
c''_n + \lambda_n c^2 c_n = q_n(t) - \frac{2c^2}{L} A(t) \quad ; \quad c_n(0) = \frac{2}{L} \int_0^L f(x) \phi_n(x) \, dx \quad ; \quad c'_n(0) = 0 \quad (44)
\]

The above IVP is very similar to the IVP considered in problem 2(a) with the exception of the last term. Define \( \tilde{q}_n(t) = q_n(t) - \frac{2c^2}{L} A(t) \). Using the same steps as the derivation of the solution for problem 2(a), the solution \( u(x,t) \) of the non-homogeneous PDE can be represented as follows

\[
u(x,t) = \sum_{n=1}^{\infty} \left[ c_n(0) \cos \sqrt{\lambda_n} c t + \int_0^t \frac{\tilde{q}_n(\tau) \sin \sqrt{\lambda_n} c (t-\tau)}{c \sqrt{\lambda_n}} \right] \cos \frac{(2n-1)\pi x}{2L}
\]

with \( c_n(0) = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} \).

4. Consider the Poisson problem

\[
u_{xx} + u_{yy} = Q(x,y), \quad 0 < x < L, \quad 0 < y < H
\]

with

\[
u_x(0,y) = u_x(L,y) = 0, \quad u(x,0) = u(x,H) = 0
\]

(a) Find the solution in terms of an eigenfunction expansion of the form

\[
u(x,y) = \sum_n A_n(y) \phi_n(x)
\]

**Solution** First, consider the solution of the related homogeneous PDE \( u_{xx} + u_{yy} = 0 \) with the given boundary conditions. Using the method of separation of variables, assume \( u(x,y) = \phi(x)g(y) \). Substituting this product form in the homogeneous PDE and simplifying, we obtain

\[
\frac{\phi''}{\phi} = \frac{g''}{g} = -\lambda
\]

(45)
where $\lambda$ is some constant. (45) results the following two ordinary differential equations. The first ODE resulting from (45) is for $\phi(x)$ and is given by $\phi'' + \lambda \phi = 0$. We now consider the boundary conditions. The boundary condition $u_x(0,y) = 0$ implies that $\phi'(0)g(y) = 0$. For nontrivial solutions, set $\phi'(0) = 0$. In an analogous manner, the boundary condition $u_x(L,y) = 0$ implies that $\phi'(L) = 0$. With this, consider the following boundary value problem.

$$\phi'' + \lambda \phi = 0 \quad (46)$$
$$\phi'(0) = 0 \quad (47)$$
$$\phi'(L) = 0 \quad (48)$$

The above boundary problem can be readily solved. The eigenvalues are given by

$$\lambda_n(x) = \left[ \frac{n\pi}{L} \right]^2 \quad n = 0, 1, 2, 3, ... \quad (49)$$

with corresponding eigenfunctions

$$\phi_n(x) = \cos \frac{n\pi x}{L} \quad (50)$$

Having determined $\lambda_n$ and $\phi_n(x)$, we now consider the non-homogeneous PDE. The starting point is the following assumption on the form of $u(x,y)$ and $Q(x,y)$.

$$u(x,y) = \sum_{n=1}^{\infty} A_n(y)\phi_n(x) \quad ; \quad Q(x,y) = \sum_{n=1}^{\infty} q_n(y)\phi_n(x) \quad (51)$$

where $q_n(y)$ is given by

$$q_n(t) = \frac{2}{L} \int_0^L Q(x,t) \cos \frac{n\pi x}{L} \, dx \quad (52)$$

Substituting the forms in (51) in the PDE, we obtain

$$\sum_{n=0}^{\infty} A_n(y)\phi''_n(x) + \sum_{n=0}^{\infty} A''_n(y)\phi_n(x) = - \sum_{n=0}^{\infty} A_n(y)\lambda_n\phi_n(x) + \sum_{n=0}^{\infty} A''_n(y)\phi_n(x) = \sum_{n=1}^{\infty} q_n(y)\phi_n(x)$$

where we have used the fact that $\phi''(x) = -\lambda_n\phi(x)$. The above equation can also be equivalently written as follows

$$\sum_{n=0}^{\infty} [A''_n - \lambda_n A_n - q_n]\phi_n(x) = 0$$

The above equation holds only if $A''_n - \lambda_n A_n - q_n = 0$. In addition, using the form of $u(x,y)$ in (51), note that $u(x,0) = \sum_{n=1}^{\infty} A_n(0)\phi_n(x) = 0$. It follows that $A_n(0) = 0$. In an analogous manner, the condition $u(x,H) = 0$ implies that $A(H) = 0$. With this, consider the following problem.

$$A''_n - \lambda_n A_n = q_n \quad ; \quad A_n(0) = 0 \quad ; \quad A_n(H) = 0 \quad (53)$$

To solve the above problem, we use variation of parameters. First, choose the homogeneous solutions as $c_1 = \sinh \sqrt{\lambda_n} y$ and $c_2 = \sinh \sqrt{\lambda_n}(H - y)$. Using variation of parameters, the particular solution is given by

$$A''_n(y) = -c_1 \int \frac{c_2 q_n}{W(c_1, c_2)} \, dy + c_2 \int \frac{c_1 q_n}{W(c_1, c_2)} \, dy \quad (54)$$
First note that
\[ W(c_1, c_2) = \det \begin{bmatrix} c_1 & c_2 \\ c_1' & c_2' \end{bmatrix} = \det \begin{bmatrix} \sinh \sqrt{\lambda_n} y & \sinh \sqrt{\lambda_n} (H - y) \\ \sqrt{\lambda_n} \cosh \sqrt{\lambda_n} y & -\sqrt{\lambda_n} \cosh \sqrt{\lambda_n} (H - y) \end{bmatrix} = -\sqrt{\lambda_n} \left[ \sinh \sqrt{\lambda_n} y \cosh \sqrt{\lambda_n} (H - y) + \cosh \sqrt{\lambda_n} y \sinh \sqrt{\lambda_n} (H - y) \right] \]

Using the characterization of the Wronskian via Abel’s theorem, noting the absence of \( A_n' \) term, it can be argued that the above Wronskian is constant. With this, choose any \( y \), for example \( y = 0 \), in the last equation above to obtain \( W(c_1, c_2) = -\sqrt{\lambda_n} \sinh \sqrt{\lambda_n} H \). Using the variation of parameters formula in (54), we obtain the following particular solution

\[
A_n^p(y) = \frac{\sinh \sqrt{\lambda_n} y}{W(c_1, c_2)} \int_y^H \sinh \sqrt{\lambda_n} (H - \tau) q_n(\tau) d\tau + \frac{\sinh \sqrt{\lambda_n} (H - y)}{W(c_1, c_2)} \int_0^y \sinh \sqrt{\lambda_n} \tau q_n(\tau) d\tau
\]

(b) Find the solution in terms of an eigenfunction expansion of the form

\[
u(x, y) = \sum_{\lambda} C_{\lambda} \Phi_{\lambda}(x, y)\]

where \( \Phi_{\lambda}(x, y) \) are suitable eigenfunctions of a Helmholtz equation.

**Solution** The associated eigenvalue problem is given by \( \nabla^2 \Phi + \lambda \Phi = 0 \). A similar problem has been solved in problem 6 of HW4 using separation of variables. Recall that the eigenvalues and the eigenfunctions are given by

\[
\lambda_{nm} = \left( \frac{n\pi}{H} \right)^2 + \left( \frac{m\pi}{L} \right)^2 \quad m = 0, 1, 2, \ldots \quad ; \quad n = 1, 2, 3, \ldots
\]

with corresponding eigenfunctions

\[
\Phi_{nm}(x, y) = \cos \left( \frac{m\pi}{L} x \right) \sin \left( \frac{n\pi}{H} y \right)
\]

Recall that the eigenfunctions \( \Phi_{nm}(x, y) \) of the Helmholtz problem are orthogonal. The starting point is the following assumption on the form of \( u(x, y) \) and \( Q(x, y) \).

\[
u(x, y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{nm} \Phi_{nm}(x, y) \quad ; \quad Q(x, y) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} q_{nm} \Phi_{nm}(x, y)
\]

where \( q_{nm} \) is given by

\[
q_{nm} = \frac{4}{LH} \int_0^L \int_0^H Q(x, y) \Phi_{nm}(x, y) \, dx \, dy
\]

Substituting the forms in (57) in the PDE, we obtain

\[
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{nm} \nabla^2 \Phi_{nm} = -\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{nm} \lambda_{nm} \Phi_{nm} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} q_{nm} \Phi_{nm}
\]

where we have used the fact that \( \nabla^2 \Phi = -\lambda \Phi \). The above equation can also be equivalently written as follows

\[
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[ \lambda_{nm} C_{nm} + q_{nm} \right] \Phi_{nm} = 0
\]
The above equation holds only if \( \lambda_{nm} C_{nm} + q_{nm} = 0 \). It follows that \( C_{nm} = -\frac{q_{nm}}{\lambda_{nm}} \). Finally, the solution \( u(x,y) \) of the non-homogeneous PDE can be represented as follows

\[
    u(x,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} -\frac{q_{nm}}{\lambda_{nm}} \cos \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi y}{H} \right)
\]
with \( q_{nm} \) given in (58).

5. Consider the boundary-value problem

\[
    (r^2 R')' - 6R = Q(r), \quad R(0) \text{ bounded, } \quad R(1) = 0
\]
where \( Q(r) \) is a smooth forcing function. Use variation of parameters to find the solution for \( R(r) \). Write the solution in the form

\[
    R(r) = \int_{0}^{1} G(r, \bar{r}) Q(\bar{r}) \, d\bar{r}
\]

Give \( G(r, \bar{r}) \) explicitly and verify (perhaps using L'Hôpital's rule) that \( R(r) \) is bounded as \( r \to 0 \).

**Solution** The associated homogeneous problem \( r^2 R'' + 2r R' - 6R = 0 \) is a Cauchy-Euler equation. Using the ansatz \( R(r) = r^n \), it can be easily checked that the two homogeneous solutions are \( r^2 \) and \( r^{-3} \). To find the particular solution of the non-homogeneous boundary value problem, we use variation of parameters. Noting the homogeneous solutions, we set the two fundamental solutions \( R_1(r) \) and \( R_2(r) \) as follows: \( R_1(r) = r^2 \) and \( R_2(r) = r^2 - r^{-3} \). Note that \( R_1(0) \) is bounded and \( R_2(1) = 0 \). Using variation of parameters, the particular solution is given by

\[
    R(r) = R_1 \underbrace{\int_{\frac{v_1}{v_1}}^{r} \frac{R_2 Q(\tau)}{W(R_1, R_2)} \, d\tau}_{v_1} + R_2 \underbrace{\int_{\frac{v_2}{v_2}}^{r} \frac{R_1 Q(\tau)}{W(R_1, R_2)} \, d\tau}_{v_2}
\]  

(59)

First note that

\[
    W(R_1, R_2) = \det \begin{bmatrix} R_1 & R_2 \\ R_1' & R_2' \end{bmatrix} = \det \begin{bmatrix} r^2 & r^2 - r^{-3} \\ 2r & 2r + 3r^{-4} \end{bmatrix} = 5r^{-2}
\]

To apply the variation of parameters formula in (59), evaluate \( v_1 \) and \( v_2 \) as follows

\[
    v_1(r) = -\int_{0}^{r} \frac{R_2 Q(\tau)}{W(R_1, R_2)} \, d\tau = -\frac{1}{5} \int_{r}^{1} [\tau^4 - \tau^{-1}] Q(\tau) \, d\tau + A
\]

\[
    v_2(r) = \int_{0}^{r} \frac{R_1 Q(\tau)}{W(R_1, R_2)} \, d\tau = \frac{1}{5} \int_{0}^{r} \tau^4 Q(\tau) \, d\tau + B
\]

where \( A \) and \( B \) are constants to be determined. Using (59), \( R(r) = r^2 v_1(r) + [r^2 - r^{-3}] v_2(r) \). The condition that \( R(0) \) is bounded implies that \( B = 0 \). The boundary condition \( R(1) = 0 \) implies that \( R(1) = A = 0 \). With this, the solution resulting from the variation of parameters formula can be written in a compact form

\[
    R(r) = \int_{0}^{1} G(r, \bar{r}) Q(\bar{r}) \, d\bar{r}
\]
with $G(r, \bar{r})$ given explicitly as follows

$$G(r, \bar{r}) = \begin{cases} \frac{1}{5} \bar{r}^4 [r^2 - r^{-3}] & \bar{r} < r \\ \frac{1}{5} r^2 [\bar{r}^4 - \bar{r}^{-1}] & \bar{r} > r \end{cases}$$

Next, we show that $R(r)$ is bounded as $r \to 0$.

$$\lim_{r \to 0^+} R(r) = \lim_{r \to 0^+} \int_r^1 \frac{1}{5} r^2 [\bar{r}^4 - \bar{r}^{-1}] Q(\bar{r}) d\bar{r}$$

$$= \lim_{r \to 0^+} \frac{1}{5} \frac{1}{r^2} \int_r^1 [\bar{r}^4 - \bar{r}^{-1}] Q(\bar{r}) d\bar{r}$$

$$= \lim_{r \to 0^+} \frac{1}{5} \frac{1}{2r^{-3}} [r^4 - r^{-1}] Q(r)$$

$$= \lim_{r \to 0^+} \frac{1}{10} [r^7 - r^2] Q(r) = 0$$

Above, the third step follows from L’Hôpital’s rule.

**Remark**: In showing that $R(r)$ is bounded as $r \to 0$, no special analysis is needed on $Q(r)$. This is because it is assumed that $Q(r)$ is a smooth varying function.