1. Text exercises 7.5.4 and 7.5.5(a), p. 290–291.

Solution

7.5.4 Substituting the product form \( u(x,y,z,t) = \phi(x,y,z)h(t) \) in the PDE, we obtain

\[
cp\phi(x,y,z)h'(t) = \nabla \cdot (K_0 \nabla \phi)h(t)
\]

Dividing both sides of the equation above by \( cp\phi(x,y,z)h(t) \) results in

\[
\frac{h'(t)}{h(t)} = \frac{1}{cp} \frac{\nabla \cdot (K_0 \nabla \phi)}{\phi} = \lambda
\]

A function of time will be equal to a function of space if both functions are equal to the same constant. Therefore,

\[
\frac{h'(t)}{h(t)} = \frac{\nabla \cdot (K_0 \nabla \phi)}{cp\phi} = -\lambda
\]

where \( \lambda \) is some constant. It can be seen that the time variable is separated. The PDE for \( \phi(x) \) resulting from the above equation is given by

\[
\nabla \cdot (K_0 \nabla \phi) + \lambda cp \phi = 0
\]

At the boundary, \( u(x,y,z) = 0 \) implies that \( \phi(x,y,z)h(t) = 0 \). For nontrivial solutions, set \( \phi(x,y,z) = 0 \) at the boundary. Therefore, \( \phi(x,y,z) \) satisfies the following eigenvalue problem

\[
\nabla \cdot (p \nabla \phi) + \lambda \sigma(x,y,z) \phi = 0 \quad ; \quad \phi = 0 \quad \text{on the boundary} \quad (1)
\]

with \( p(x,y,z) = K_0 \) and \( \sigma(x,y,z) = cp \).

7.5.5 Define the operator \( L = \nabla \cdot (p \nabla \phi) \). To derive the three dimensional Green’s formula, we first note that \( uL(v) - vL(u) = u\nabla \cdot (p \nabla v) - v\nabla \cdot (p \nabla u) \). Recall the identity \( \nabla \cdot (aB) = \nabla a \cdot B + a \nabla \cdot B \) which holds for a scalar valued function \( a \) and a vector \( B \). Using this identity, we have the following equations.

\[
\nabla \cdot (up \nabla v) = \nabla u \cdot p \nabla v + u \nabla \cdot (p \nabla v)
\]
\[
\nabla \cdot (vp \nabla u) = \nabla v \cdot p \nabla u + v \nabla \cdot (p \nabla u)
\]

Subtracting the second equation from the first, we obtain

\[
u \nabla \cdot (p \nabla v) - v \nabla \cdot (p \nabla u) = \nabla \cdot (up \nabla v) - \nabla \cdot (vp \nabla u)
\]

Integrating both sides, we have

\[
\int \int \int_R u \nabla \cdot (p \nabla v) - v \nabla \cdot (p \nabla u) \, dx \, dy \, dz = \int \int \int_R \nabla \cdot (up \nabla v - vp \nabla u) \, dx \, dy \, dz
\]
\[
= \oint (up \nabla v - vp \nabla u) \cdot n \, dS
\]
with the last equality following from divergence theorem. In summary, we have the following result which is analogous to the result (see 7.5.8 and 7.5.9) in the text book.

\[
\int \int \int_R (uLv - vLu) \, dx \, dy \, dz = \int \int \int_R (u \nabla \cdot (p \nabla v) - v \nabla \cdot (p \nabla u)) \, dx \, dy \, dz = 0
\]

Using this result, we now proceed to show that eigenfunctions belonging to different eigenvalues are orthogonal with some weight to be determined. Let \( \phi_n(x, y, z) \) and \( \phi_m(x, y, z) \) be eigenfunctions with corresponding eigenvalues \( \lambda_n \) and \( \lambda_m \) respectively. From the boundary condition of the eigenvalue problem, \( \phi_n(x, y, z) = \phi_m(x, y, z) = 0 \) on the boundary. With this, setting \( u = \phi_n(x, y, z) \) and \( v = \phi_m(x, y, z) \) in the above main result, we obtain

\[
\int \int \int_R \phi_n L \phi_m - \phi_m L \phi_n \, dx \, dy \, dz = 0
\]

With this, the assumption in the main result is satisfied and applying the three dimensional Green’s formula gives

\[
\int \int \int_R \phi_n L \phi_m - \phi_m L \phi_n \, dx \, dy \, dz = \int \int \int_R \phi_n (-\lambda_m \sigma \phi_m) + \phi_m (\lambda_n \sigma \phi_n) \, dx \, dy \, dz
\]

\[
= (\lambda_n - \lambda_m) \int \int \int_R \phi_m \phi_n \sigma \, dx \, dy \, dz = 0
\]

Since \( \lambda_m \neq \lambda_n \), it follows that \( \int \int \int_R \phi_m \phi_n \sigma \, dx \, dy \, dz = 0 \). Therefore, eigenfunctions belonging to different eigenvalues are orthogonal with weight \( \sigma(x, y, z) \).

2. Consider the heat flow in a three-dimensional region \( \Omega \) with smooth boundary \( \partial \Omega \) and unit normal \( \mathbf{n} \). The temperature \( u(x, t) \) satisfies

\[
\frac{\partial u}{\partial t} = k \nabla^2 u + Q(x), \quad x \in \Omega, \quad t > 0,
\]

where \( k \) is a constant thermal diffusivity and \( Q(x) \) is a heat source. The initial condition is \( u(x, 0) = f(x) \) and the boundary condition is \( \mathbf{n} \cdot \nabla u = g(x) \) for \( x \in \partial \Omega \). Use the divergence theorem to determine a condition on \( Q \) and \( g \) such that a steady state solution exists. (You need not find this solution.) Give a brief physical interpretation of the condition you derive in terms of heat generation and heat flux. [Hint: the interpretation is a generalization of one discussed back in Chapter 1 of the text.]

**Solution** At steady state, \( u_t = 0 \) implying that \( k \nabla^2 u + Q(x) = 0 \). A volume integral of this equation results

\[
\int_\Omega [(k \nabla^2 u + Q(x)) \, dv = \int_\Omega k \nabla^2 u \, dv + \int_\Omega Q(x) \, dv = 0
\]

Applying the divergence theorem, we obtain

\[
\int_\Omega k \nabla^2 u \, dv + \int_\Omega Q(x) \, dv = \int_{\partial \Omega} k \nabla u \cdot \mathbf{n} \, dS + \int_\Omega Q(x) \, dv = 0
\]
Using the boundary condition \( \mathbf{n} \cdot \nabla u = g(\mathbf{x}) \) for \( \mathbf{x} \in \partial \Omega \) in the above equation, we have

\[
\int_{\partial \Omega} kg(\mathbf{x}) \, dS + \int_{\Omega} Q(\mathbf{x}) \, dv = 0
\]  

(2)

The above equation has the following physical interpretation. At the steady state, the total heat generated inside the surface is equal to the total heat flux at the boundary.

**3.** Solve \( u_{tt} = c^2 \nabla^2 u \) for \( t > 0 \) and \( \mathbf{x} \in \Omega \), where \( \Omega \) is a circular disk of radius \( a \), subject to \( u = 0 \) on the boundary of \( \Omega \) and the initial conditions

\[ u(r, \theta, 0) = 0, \quad u_t(r, \theta, 0) = \beta(r) \sin(2\theta). \]

[Hint: It may be helpful to consider solutions of the form \( u(r, \theta, t) = v(r, t) \sin(2\theta) \).] Sketch contours of representative modes of vibration for this problem.

**Solution** Using the method of separation of variables, assume \( u(r, \theta, t) = v(r, t) \sin(2\theta) \). Substituting this product form in the PDE, we obtain

\[
v_{tt} \sin(2\theta) = c^2 \left[ \frac{1}{r} (rv_r)_r \sin(2\theta) - \frac{4}{r^2} v \sin(2\theta) \right]
\]

Dividing both sides of the equation above by \( \sin(2\theta) \) results

\[
v_{tt} = c^2 \left[ \frac{1}{r} (rv_r)_r - \frac{4}{r^2} v \right]
\]

To solve the above PDE, once again using separation of variables, assume the form \( v(r, t) = R(r)T(t) \) and substitute in the above equation to obtain

\[
R(r)T''(t) = c^2 \left[ \frac{1}{r} (rR')' T(t) - \frac{4}{r^2} R(r)T(t) \right]
\]

Dividing both sides of the equation above by \( c^2 R(r)T(t) \) results

\[
\frac{T''(t)}{c^2 T(t)} = \frac{1}{Rr} (rR')' - \frac{4}{r^2}
\]

A function of time will be equal to a function of space if both functions are equal to the same constant. Therefore,

\[
\frac{T''(t)}{c^2 T(t)} = \frac{1}{Rr} (rR')' - \frac{4}{r^2} = -\lambda
\]

(3)

where \( \lambda \) is some constant. (3) results two ordinary differential equations. The first ODE resulting from (3) is for \( R(r) \) and is given by

\[
\frac{1}{Rr} (rR')' - \frac{4}{r^2} = -\lambda.
\]

Rearranging the terms results \( r(rR')' + (\lambda r^2 - 4)R = 0 \). We now consider the boundary conditions. The condition \( u = 0 \) on the boundary of \( \Omega \) implies that \( u(a, \theta, t) = v(a, t) \sin(2\theta) = R(a)T(t) \sin(2\theta) = 0 \). For nontrivial solutions, set \( R(a) = 0 \) on the boundary. In addition, for the solution to be bounded at \( r = 0 \), we require that \( |R(0)| < \infty \). With this, consider the following boundary value problem.

\[
r^2 R'' + rR' + (\lambda r^2 - 4)R = 0
\]

\[
R(a) = 0
\]

\[
|R(0)| < \infty
\]
Then, by the now familiar transformation \( z = \sqrt{\lambda}r \), the boundary value problem can be represented as follows.

\[
\begin{align*}
z^2 R'' + z R' + (z^2 - 4)R &= 0 \\
R(\sqrt{\lambda}a) &= 0 \\
|R(0)| &< \infty
\end{align*}
\]  

(4) is Bessel’s differential equation of order 2. The general solution of (4) is given by \( R(z) = c_1J_2(z) + c_2 Y_2(z) \) where \( J_2(z) \) is Bessel’s equation of the first kind of order 2 and \( Y_2(z) \) is Bessel’s equation of the second kind of order 2. Applying the second boundary condition in (6), \(|R(0)| < \infty\) implies that \( c_2 = 0 \). Therefore, \( R(z) = c_1J_2(z) \). Applying the first boundary condition in (5), \( R(\sqrt{\lambda}a) = c_1J_2(\sqrt{\lambda}a) = 0 \).

For nontrivial solutions, set \( \lambda_n = (\frac{2n}{a})^2 \). In summary, the eigenvalues are given by

\[
\lambda_n = \left( \frac{2n}{a} \right)^2
\]

with corresponding eigenfunctions expressed in the original coordinate \( r \) as follows

\[
R_n(r) = c_n J_2(\sqrt{\lambda_n}r) = c_n J_2(\frac{2n}{a}r)
\]

The second ODE resulting from (3) is for \( T(t) \) and is given by \( T''(t) + \lambda c^2 T(t) = 0 \). Noting the form of \( \lambda_n \) in (8), the general solution for the ODE has the following form.

\[
T_n(t) = a_n \cos \left( \frac{2n}{a}ct \right) + b_n \sin \left( \frac{2n}{a}ct \right)
\]

where \( a_n \) and \( b_n \) are constants. The initial condition \( u(r, \theta, 0) = 0 \) implies \( R(r) T(0) \sin(2\theta) = 0 \). For nontrivial solutions, set \( T(0) = 0 \). Using this condition in (9), \( T(0) = a_n = 0 \). With this, it follows that

\[
T_n(t) = b_n \sin \left( \frac{2n}{a}ct \right)
\]

Using the principle of superposition, the solution to the PDE can be written as follows.

\[
u(r, \theta, t) = \sum_{n=1}^{\infty} A_n J_2 \left( \frac{2n}{a}r \right) \sin \left( \frac{2n}{a}ct \right) \sin(2\theta)
\]

(10) where \( A_n \) is some constant. Finally, apply the second initial condition.

\[
u_t(r, \theta, 0) = \sum_{n=1}^{\infty} A_n \frac{2n}{a} c J_2 \left( \frac{2n}{a}r \right) \sin(2\theta) = \beta(r) \sin(2\theta)
\]

Recall that the Bessel functions are orthogonal with weight \( r \). With this, \( A_n \) is given by

\[
A_n = \frac{a}{\frac{a}{2n} c} \left( \int_0^a \beta(r) J_2 \left( \frac{2n}{a}r \right) r \, dr \right) \left( \int_0^a \left[ J_2 \left( \frac{2n}{a}r \right) \right]^2 r \, dr \right)
\]

With this, using (10), the solution to the PDE has the following form.

\[
u(r, \theta, t) = \sum_{n=1}^{\infty} \frac{a}{\frac{a}{2n} c} \left( \int_0^a \beta(r) J_2 \left( \frac{2n}{a}r \right) r \, dr \right) \left( \int_0^a \left[ J_2 \left( \frac{2n}{a}r \right) \right]^2 r \, dr \right) J_2 \left( \frac{2n}{a}r \right) \sin \left( \frac{2n}{a}ct \right) \sin(2\theta)
\]

Figures 1-2 below show contours of some of the eigenmodes. We have set \( a = 1 \) and the zeros of \( J_2(z) \) are computed using the \texttt{fzero} command in Matlab.
4. Consider the following two eigenvalues problems. For each case, find a formula of the form $F(\lambda) = 0$ that determines the eigenvalues $\lambda$, and find the corresponding eigenfunctions $\phi(r)$. Use Maple’s BesselJ and BesselY functions (or Matlab’s besselj and bessely) to determine numerical values for the smallest three eigenvalues for each problem, and plot the corresponding eigenfunctions.

(a) $r^2 \phi'' + r \phi' + (\lambda r^2 - 1) \phi = 0, \quad 0 < r < 2, \quad \phi(0) \text{ bounded and } \phi(2) = 0$

(b) $r^2 \phi'' + r \phi' + (\lambda r^2 - 4) \phi = 0, \quad 1 < r < 3, \quad \phi'(1) = 0 \text{ and } \phi(3) = 0$

Solution

(a) The general solution of the Bessel ODE is given by

$$\phi(r) = c_1 J_1(\sqrt{\lambda}r) + c_2 Y_1(\sqrt{\lambda}r)$$

where $c_1$ and $c_2$ are some constants. The first boundary condition that $\phi(0)$ is bounded implies that $c_2 = 0$. Therefore, $\phi(r) = c_1 J_1(\sqrt{\lambda}r)$. Applying the second boundary
condition, $\phi(2) = c_1 J_1(2\sqrt{\lambda}) = 0$. For nontrivial solutions, set $J_1(2\sqrt{\lambda}) = 0$ with $\lambda \neq 0$. Let $z_{1n}$ denote the $n$-th zero of $J_1$. With this, the eigenvalues are given by

$$\lambda_n = \frac{z_{1n}^2}{4}$$

with corresponding eigenfunctions

$$\phi_n(r) = J_1 \left( \frac{z_{1n} r}{2} \right)$$

The zeros of $J_1(2\sqrt{\lambda})$ can be computed using the `fzero` command in Matlab. With this, the first three eigenvalues are

$$\lambda_1 = 3.6705 \; ; \; \lambda_2 = 12.3046 \; ; \; \lambda_3 = 25.8749$$

with corresponding eigenfunctions

$$\phi_1(r) = J_1(1.9159r) \; ; \; \phi_2(r) = J_1(3.5078r) \; ; \; \phi_3(r) = J_1(5.0867r)$$

Figure 3 shows the first three eigenfunctions.

![Figure 3: Plot of the first three eigenfunctions](image)

(b) The general solution of the Bessel ODE is given by

$$\phi(r) = c_1 J_2(\sqrt{\lambda}r) + c_2 Y_2(\sqrt{\lambda}r)$$

where $c_1$ and $c_2$ are constants. Applying the second boundary condition, $\phi(3) = c_1 J_2(3\sqrt{\lambda}) + c_2 Y_2(3\sqrt{\lambda}) = 0$. Set $c_1 = k Y_2(3\sqrt{\lambda})$ and $c_2 = -k J_2(3\sqrt{\lambda})$ where $k$ is some constant. It follows that

$$\phi(r) = k Y_2(3\sqrt{\lambda}) J_2(\sqrt{\lambda}r) - k J_2(3\sqrt{\lambda}) Y_2(\sqrt{\lambda}r)$$

The second boundary condition requires a form of $\phi'(r)$ which can be evaluated as follows.

$$\phi'(r) = k \sqrt{\lambda} \left( Y_2(3\sqrt{\lambda}) J'_2(\sqrt{\lambda}r) - J_2(3\sqrt{\lambda}) Y'_2(\sqrt{\lambda}r) \right)$$
Applying the second boundary condition, we have
\[ \phi'(1) = k\sqrt{\lambda} \left( Y_2(3\sqrt{\lambda})J'_2(\sqrt{\lambda}) - J_2(3\sqrt{\lambda})Y'_2(\sqrt{\lambda}) \right) = 0 \]

For nontrivial solutions, \( k \neq 0 \). In addition, since \( Y_2(z) \) is singular at \( z = 0 \), \( \lambda \neq 0 \). Define the function \( F(p) \) as follows
\[ F(p) = Y_2(3p)J'_2(p) - J_2(3p)Y'_2(p) \]

Let \( p_n \) denote the zeros of \( F(p) \). For the second boundary condition to hold, we require \( \sqrt{\lambda_n} = p_n \). With this, the eigenvalues are given by \( \lambda_n = p_n^2 \) with corresponding eigenfunctions
\[ \phi_n(r) = Y_2(3p_n)J_2(p_nr) - J_2(3p_n)Y_2(p_nr) \]

The zeros of \( F(p) \) can be computed using the \textit{fzero} command in Matlab. With this, the first three eigenvalues are
\[ \lambda_1 = 2.5384 \; ; \; \lambda_2 = 7.5186 \; ; \; \lambda_3 = 17.2709 \]

with corresponding eigenfunctions
\[
\begin{align*}
\phi_1(r) &= Y_2(3p_1)J_2(p_1r) - J_2(3p_1)Y_2(p_1r) \; ; \; p_1 = 1.5932 \\
\phi_2(r) &= Y_2(3p_2)J_2(p_2r) - J_2(3p_2)Y_2(p_2r) \; ; \; p_2 = 2.7420 \\
\phi_3(r) &= Y_2(3p_3)J_2(p_3r) - J_2(3p_3)Y_2(p_3r) \; ; \; p_3 = 4.1558
\end{align*}
\]

Figure 4 shows the first three eigenfunctions.

**Figure 4: Plot of the first three eigenfunctions**

5. Solve \( \nabla^2 u = 0 \) in the half-cylinder \( 0 < r < a, \; 0 < \theta < \pi, \; 0 < z < H \) subject to the boundary conditions
\[
\begin{align*}
u(a, \theta, z) &= 0, \quad u_\theta(r, 0, z) = u_\theta(r, \pi, z) = 0, \\
u_z(r, \theta, 0) &= 0, \quad u(r, \theta, H) = \beta(r) \cos \theta.
\end{align*}
\]
[Hint: It may be helpful to consider solutions of the form \( u(r, \theta, z) = w(r, z) \cos \theta \).]

**Solution** Using the method of separation of variables, assume \( u(r, \theta, z) = w(r, z) \cos(\theta) \). Substituting this product form in the PDE, we obtain

\[
\frac{1}{r} (rw_r)_r + \frac{1}{r^2} w \cos(\theta) + w_{zz} \cos(\theta) = 0
\]

Dividing both sides of the equation above by \( \cos(\theta) \) results in

\[
\frac{1}{r} (rw_r)_r - \frac{1}{r^2} w + w_{zz} = 0
\]

To solve the above PDE, once again using separation of variables, assume the form \( w(r, z) = f(r)h(z) \) and substitute in the above equation to obtain

\[
\frac{1}{r} (rf')' h(z) - \frac{1}{r^2} f(r)h(z) + h''f(r) = 0
\]

Dividing both sides of the equation above by \( f(r)h(z) \) results in

\[
\frac{1}{r} (rf')' - \frac{1}{r^2} = -\frac{h''}{h} = -\lambda
\]

A function of \( z \) will be equal to a function of \( r \) if both functions are equal to the same constant. Therefore,

\[
\frac{1}{r} (rf')' - \frac{1}{r^2} = -\frac{h''}{h} = -\lambda
\]

where \( \lambda \) is some constant. (11) results two ordinary differential equations. The first ODE resulting from (11) is for \( f(r) \) and is given by \( \frac{1}{r} (rf')' - \frac{1}{r^2} = -\lambda \). Rearranging the terms results in \( r (rf')' + (\lambda r^2 - 1) f = 0 \). We now consider the boundary conditions. The condition \( u(a, \theta, z) = 0 \) implies that \( u(a, \theta, z) = w(a, z) \cos(\theta) = f(a)h(z) \cos(\theta) = 0 \). For nontrivial solutions, set \( f(a) = 0 \). In addition, for the solution to be bounded at \( r = 0 \), we require that \( |f(0)| < \infty \). With this, consider the following boundary value problem.

\[
r^2 f'' + rf' + (\lambda r^2 - 1) f = 0 \quad (12)
f(a) = 0 \quad (13)
|f(0)| < \infty \quad (14)
\]

The general solution of (12) is given by \( f(r) = c_1 J_1(\sqrt{\lambda}r) + c_2 Y_1(\sqrt{\lambda}r) \) where \( J_1 \) is Bessel’s equation of the first kind of order 1 and \( Y_1 \) is Bessel’s equation of the second kind of order 1. Applying the second boundary condition in (14), \( |f(0)| < \infty \) implies that \( c_2 = 0 \). Therefore, \( f(r) = c_1 J_1(\sqrt{\lambda}r) \). Applying the first boundary condition in (13), \( f(a) = c_1 J_1(\sqrt{\lambda}a) = 0 \). For nontrivial solutions, set \( J_1(\sqrt{\lambda}a) = 0 \). Let \( z_{1n} \) denote the \( n \)-th zero of \( J_1 \). With this, we have \( \lambda_n = \left( \frac{z_{1n}}{a} \right)^2 \). In summary, the eigenvalues are given by

\[
\lambda_n = \left( \frac{z_{1n}}{a} \right)^2 \quad (15)
\]

with corresponding eigenfunctions

\[
R_n(r) = c_n J_1(\sqrt{\lambda_n}r) = c_n J_1 \left( \frac{z_{1n}}{a} r \right) \quad (16)
\]
The second ODE resulting from (11) is for \( h(z) \) and is given by \( h''(z) - \lambda h(z) = 0 \). Noting the form of \( \lambda_n \) in (16), the general solution for the ODE has the following form.

\[
h_n(z) = a_n \cosh \left( \frac{z_{1n}}{a} z \right) + b_n \sinh \left( \frac{w_{1n}}{a} z \right)
\]  

(17)

where \( a_n \) and \( b_n \) are some constants. The initial condition \( u_z(r, \theta, 0) = 0 \) implies \( f(r)h'(0)\cos(\theta) = 0 \). For nontrivial solutions, set \( h'(0) = 0 \). Using this condition in (17), noting that \( h'(0) = 0 \) implies that \( b_n = 0 \), we have

\[
h_n(z) = b_n \cosh \left( \frac{z_{1n}}{a} z \right)
\]

Using the principle of superposition, the solution to the PDE can be written as follows.

\[
u(r, \theta, t) = \left[ \sum_{n=1}^{\infty} A_n J_1 \left( \frac{z_{1n}}{a} r \right) \cosh \left( \frac{z_{1n}}{a} z \right) \right] \cos(\theta)
\]  

(18)

where \( A_n \) is some constant. Finally, apply the initial condition \( u(r, \theta, H) = \beta(r) \cos(\theta) \).

\[
u(r, \theta, H) = \left[ \sum_{n=1}^{\infty} A_n J_1 \left( \frac{z_{1n}}{a} r \right) \cosh \left( \frac{z_{1n}}{a} H \right) \right] \cos(\theta) = \beta(r) \cos(\theta)
\]

Recall that the Bessel functions are orthogonal with weight \( r \). With this, \( A_n \) is given by

\[
A_n = \frac{1}{\cosh \left( \frac{z_{1n}}{a} H \right)} \frac{\int_0^a \beta(r) J_1 \left( \frac{z_{1n}}{a} r \right) r dr}{\int_0^a \left[ J_1 \left( \frac{z_{1n}}{a} r \right) \right]^2 r dr}.
\]

With this, using (18), the solution to the PDE has the following form.

\[
u(r, \theta, z) = \left[ \sum_{n=1}^{\infty} \frac{1}{\cosh \left( \frac{z_{1n}}{a} H \right)} \frac{\int_0^a \beta(r) J_1 \left( \frac{z_{1n}}{a} r \right) r dr}{\int_0^a \left[ J_1 \left( \frac{z_{1n}}{a} r \right) \right]^2 r dr} J_1 \left( \frac{z_{1n}}{a} r \right) \cosh \left( \frac{z_{1n}}{a} z \right) \right] \cos(\theta)
\]

**Remark:** Note that the separation of variable assumption \( u(r, \theta, z) = w(r, z) \cos \theta \) naturally enforces the boundary conditions \( u_\theta(r, 0, z) = u_\theta(r, \pi, z) = 0 \).