1. Text exercise 5.3.9, p. 163. (Note: The ODE is this problem is a Cauchy-Euler equation and has solutions of the form \( \phi(x) = x^k \), where \( k \) is a constant, possibly complex.)

Solution

(a) Multiplying by \( \frac{1}{x^2} \), we obtain

\[
\frac{d^2 \phi}{dx^2} + \frac{d\phi}{dx} \frac{\lambda}{x} + \phi = 0
\]

The equation above can also equivalently be written as

\[
\frac{d}{dx} \left( x \frac{d\phi}{dx} \right) + \frac{\lambda}{x} \phi = 0
\]

This is a Sturm-Liouville ODE with \( p = x \), \( q = 0 \) and \( \sigma = 1/x \). Note, in \( x \in [1, b] \), \( p > 0 \) and \( \sigma > 0 \) in \([1, b]\). Also, \( p, q \) and \( \sigma \) are continuous in \([1, b]\). Since the boundary conditions satisfy the Sturm-Liouville form, it is a regular Sturm-Liouville ODE.

(b) Using the Rayleigh quotient, we have

\[
\lambda = \frac{-p \phi \left. \frac{d\phi}{dx} \right|_a^b + \int_a^b \left[ p \left( \frac{d\phi}{dx} \right)^2 - q \phi^2 \right] dx}{\int_a^b \phi^2 \sigma dx}
\]

\begin{align*}
= & \frac{-x \phi \left. \frac{d\phi}{dx} \right|_1^b + \int_1^b \left( x \frac{d\phi}{dx} \right)^2 dx}{\int_1^b \phi^2 \frac{1}{x} dx} \\
= & \frac{\int_1^b \left( x \frac{d\phi}{dx} \right)^2 dx}{\int_1^b \phi^2 \frac{1}{x} dx}
\end{align*}

Note that \( x > 0 \) for \( 1 \leq x \leq b \) and \( \frac{1}{x} > 0 \) for \( 1 \leq x \leq b \). Also, the terms \( \left( x \frac{d\phi}{dx} \right)^2 \) and \( \phi^2 \) are always nonnegative. It follows that \( \lambda \geq 0 \).

(c) We consider two cases \( \lambda = 0 \) and \( \lambda > 0 \). The case \( \lambda < 0 \) is not necessary since we have established in (b) that all eigenvalues are nonnegative.

\( \lambda = 0 \): The ODE is a Cauchy-Euler equation. For \( \lambda = 0 \), the general solution is given by

\[
\phi(x) = c_1 + c_2 \ln x
\]

Applying the first boundary condition, \( \phi(1) = c_1 = 0 \). Hence, \( \phi(x) = c_2 \ln x \). Applying the second boundary condition, \( \phi(b) = c_2 \ln b = 0 \). Since \( b > 1 \), \( c_2 = 0 \). Therefore, \( \phi(x) = 0 \) is the trivial solution. This implies that there is no nontrivial solution for \( \lambda = 0 \).

\( \lambda > 0 \): The ODE is a Cauchy-Euler equation. For \( \lambda = 0 \), the general solution is given by

\[
\phi(x) = c_1 \cos \left( \sqrt{\lambda} \ln x \right) + c_2 \sin \left( \sqrt{\lambda} \ln x \right)
\]
Applying the first boundary condition, $\phi(1) = c_1 = 0$. Hence, $\phi(x) = c_2 \sin \left( \sqrt{\lambda} \ln x \right)$.

Applying the second boundary condition, $\phi(b) = c_2 \sin \left( \sqrt{\lambda} \ln b \right) = 0$. For nontrivial solutions, set $\sqrt{\lambda} \ln b = n\pi$ where $n$ is a positive integer. Therefore, the eigenvalues are given by

$$\lambda_n = \left( \frac{n\pi}{\ln b} \right)^2 ; \quad n = 1, 2, 3, ...$$

From the above equation, it can be seen that there is an infinite number of eigenvalues and the smallest eigenvalue is given by

$$\lambda_1 = \left( \frac{\pi}{\ln b} \right)^2$$

The eigenfunctions are given by

$$\phi_n(x) = \sin \left( \frac{n\pi}{\ln b} \ln x \right) ; \quad n = 1, 2, 3, ...$$

(d) According to Sturm-Liouville theory, the eigenfunctions are orthogonal with weight $\sigma(x) = \frac{1}{x}$. With this, we check if the eigenfunctions are orthogonal. First, we have

$$\int_{1}^{b} \sin \left( \frac{n\pi}{\ln b} \ln x \right) \sin \left( \frac{m\pi}{\ln b} \ln x \right) \frac{1}{x} dx$$

We do a $u$ substitution with $u = \frac{1}{x}$ in the above integral to obtain

$$\int_{0}^{\ln b} \sin \left( \frac{n\pi}{\ln b} u \right) \sin \left( \frac{m\pi}{\ln b} u \right) du = \frac{1}{2} \int_{-\ln b}^{\ln b} \sin \left( \frac{n\pi}{\ln b} u \right) \sin \left( \frac{m\pi}{\ln b} u \right) du$$

Recalling the orthogonality relation for sine, it can be concluded that

$$\frac{1}{2} \int_{-\ln b}^{\ln b} \sin \left( \frac{n\pi}{\ln b} u \right) \sin \left( \frac{m\pi}{\ln b} u \right) du = \begin{cases} 0 & n \neq m \\ \ln b & n = m \end{cases}$$

Therefore, the eigenfunctions are orthogonal with weight $\frac{1}{x}$.

(e) With $\phi_n(x) = \sin \left( \frac{n\pi}{\ln b} \ln x \right)$, let $z_n = \frac{n\pi}{\ln b} \ln x$. Note that, at $x = 1$, $z_n = 0$ and at $x = b$, $z_n = n\pi$. Also, observe that $\sin(z_n)|_{z_n=0} = 0$ and $\sin(z_n)|_{z_n=n\pi} = 0$. With these observations, it can be concluded that the $n$-th eigenfunction has $n-1$ zeros.

Illustrating with examples, for $n = 1$, the domain of $\phi_1(x)$ is between 0 and $\pi$. There are no zeros for this eigenfunction since end points are not considered. For $n = 2$, the domain of $\phi_2(x)$ is between 0 and $2\pi$. There is 1 zero for this eigenfunction since end points are not considered. This argument can be applied for all $n$ and it follows that $\phi_n(x)$ has $n-1$ zeros.
2. Text exercise 5.4.5, p. 167.

**Solution** Using the method of separation of variables, assume \( u(x, t) = \phi(x)h(t) \). Substituting this product form in the PDE, we obtain

\[
\rho(x)\phi(x)h''(t) = T_0 \phi''(x)h(t) + \alpha(x)\phi(x)h(t)
\]

Dividing both sides of the equation above by \( \rho(x)\phi(x)h(t) \) results in

\[
\frac{h''(t)}{h(t)} = \frac{T_0}{\rho(x)} \frac{\phi''(x)}{\phi(x)} + \frac{\alpha(x)}{\rho(x)} = -\lambda
\]

where \( \lambda \) is some constant. (1) results the following two ordinary differential equations. The first ODE resulting from (1) is for \( \phi(x) \) and is given by

\[
T_0 \phi''(x) + \alpha(x)\phi(x) + \lambda \rho(x)\phi(x) = 0.
\]

We now consider the boundary conditions. The first boundary condition \( u(0,t) = 0 \) implies that \( \phi(0)h(t) = 0 \). For nontrivial solutions, set \( \phi(0) = 0 \). In an analogous manner, the second boundary condition \( u(L,t) = 0 \) implies that \( \phi(L) = 0 \). With this, consider the following boundary value problem.

\[
T_0 \phi''(x) + \alpha(x)\phi(x) + \lambda \rho(x)\phi(x) = 0 \quad (2)
\]

\[
\phi(0) = 0 \quad (3)
\]

\[
\phi(L) = 0 \quad (4)
\]

Assuming that \( T_0 \) is a positive constant, (2) is a Sturm-Liouville ODE with \( p = T_0 \), \( q = \alpha \) and \( \sigma = \rho \). Since the boundary conditions satisfy the Sturm-Liouville form, it is a regular Sturm-Liouville ODE. Using the Rayleigh quotient, we have

\[
\lambda = \frac{-p\phi \frac{d\phi}{dx} \big|_a^b + \int_a^b \left[ p \left( \frac{d\phi}{dx} \right)^2 - q\phi^2 \right] dx}{\int_a^b \phi^2 \sigma dx} = \frac{-T_0\phi \frac{d\phi}{dx} \big|_0^L + \int_0^L \left[ T_0 \left( \frac{d\phi}{dx} \right)^2 - \alpha(x)\phi^2 \right] dx}{\int_0^L \phi^2 \rho(x) dx} = \frac{\int_0^L \left[ T_0 \left( \frac{d\phi}{dx} \right)^2 - \alpha(x)\phi^2 \right] dx}{\int_0^L \phi^2 \rho(x) dx}
\]

By assumption, \( \alpha(x) < 0 \) and \( \rho(x) > 0 \). In addition, the terms \( \left( \frac{d\phi}{dx} \right)^2 \) and \( \phi^2 \) are always nonnegative. It follows that \( \lambda \geq 0 \). In fact, \( \lambda > 0 \). Noting the Rayleigh quotient relation, the only way \( \lambda = 0 \) is if \( \phi(x) = 0 \) and \( \phi'(x) = 0 \). This leads to \( \phi(x) = 0 \) which is a trivial solution. In summary, using Sturm-Liouville theory, there is an infinite number of eigenvalues

\[
0 < \lambda_1, \lambda_2, ..., \lambda_n < \lambda_{n+1} < ...
\]
Corresponding to each eigenvalue \( \lambda_n \), there is an eigenfunction \( \phi_n(x) \). Sturm-Liouville theory also informs that the eigenfunctions form a complete orthogonal set. With this, we complete the analysis of the first ODE resulting from (1). The second ODE resulting from (1) is for \( h(t) \) and is given by \( h''(t) + \lambda h(t) = 0 \). This is a constant coefficient second order linear ODE. The general solution for the ODE has the following form.

\[
h(t) = c_1 \cos \left( \sqrt{\lambda} \, t \right) + c_2 \sin \left( \sqrt{\lambda} \, t \right)
\]

where \( c_1 \) and \( c_2 \) are some constants. Using the principle of superposition, the solution to the PDE can be written as follows.

\[
u(x,t) = \sum_{n=1}^{\infty} \left[ A_n \cos \left( \sqrt{\lambda_n} \, t \right) + B_n \sin \left( \sqrt{\lambda_n} \, t \right) \right] \phi_n(x)
\]

First, apply the initial condition \( u(x,0) = f(x) \) to obtain

\[
u(x,0) = \sum_{n=1}^{\infty} A_n \phi_n(x) = f(x)
\]

From this, it follows that

\[
A_n = \frac{\int_0^L f(x) \phi_n(x) \rho(x) \, dx}{\int_0^L \phi_n(x)^2 \rho(x) \, dx}
\]

Applying the second initial condition \( u_t(x,0) = g(x) \), we have

\[
u(x,0) = \sum_{n=1}^{\infty} B_n \sqrt{\lambda_n} \phi_n(x) = g(x)
\]

From this, it follows that

\[
B_n = \frac{1}{\sqrt{\lambda_n}} \frac{\int_0^L g(x) \phi_n(x) \rho(x) \, dx}{\int_0^L \phi_n(x)^2 \rho(x) \, dx}
\]

Having determined the coefficients, the solution to the PDE has the following explicit form

\[
u(x,t) = \sum_{n=1}^{\infty} \left[ A_n \cos \left( \sqrt{\lambda_n} \, t \right) + B_n \sin \left( \sqrt{\lambda_n} \, t \right) \right] \phi_n(x)
\]

with \( A_n \) and \( B_n \) given in (7) and (8) respectively.

3. Text exercise 5.5.1, p. 174, parts (a), (b), (c) and (d) only. 

Solution

(a) With \( u(0) = v(0) = 0 \) and \( u(L) = v(L) = 0 \), consider

\[
p \left( \frac{dv}{dx} - v \frac{du}{dx} \right)^L_0.
\]

\[
p \left( \frac{dv}{dx} - v \frac{du}{dx} \right)^L_0 = p(L) \left( u(L) \frac{dv}{dx} - u(L) \frac{du}{dx} \right) - p(0) \left( u(0) \frac{dv}{dx} - u(0) \frac{du}{dx} \right)
\]

\[
= 0
\]

Therefore, the boundary conditions yield a self-adjoint problem.
(b) With \( \frac{du}{dx}(0) = \frac{dv}{dx}(0) = 0 \) and \( u(L) = v(L) = 0 \), consider \( p \left( \frac{dv}{dx} - \frac{du}{dx} \right) \bigg|_0^L \).

\[
p \left( \frac{dv}{dx} - \frac{du}{dx} \right) \bigg|_0^L = p(L) \left( u(L) \frac{dv}{dx}(L) - v(L) \frac{du}{dx}(L) \right) - p(0) \left( u(0) \frac{dv}{dx}(0) - v(0) \frac{du}{dx}(0) \right) = 0
\]

Therefore, the boundary conditions yield a self-adjoint problem.

(c) With \( \frac{du}{dx}(0) - hu(0) = \frac{dv}{dx}(0) - hv(0) = 0 \) and \( \frac{du}{dx}(L) = \frac{dv}{dx}(L) = 0 \), consider \( p \left( \frac{dv}{dx} - \frac{du}{dx} \right) \bigg|_0^L \).

\[
p \left( \frac{dv}{dx} - \frac{du}{dx} \right) \bigg|_0^L = p(L) \left( u(L) \frac{dv}{dx}(L) - v(L) \frac{du}{dx}(L) \right) - p(0) \left( u(0) \frac{dv}{dx}(0) - v(0) \frac{du}{dx}(0) \right) = -p(0) \left[ u(0)hv(0) - v(0)hu(0) \right] = 0
\]

Therefore, the boundary conditions yield a self-adjoint problem.

(d) With \( u(a) = u(b) \), \( v(a) = v(b) \), \( p(a) \frac{du}{dx}(a) = p(b) \frac{du}{dx}(b) \) and \( p(a) \frac{dv}{dx}(a) = p(b) \frac{dv}{dx}(b) \), consider \( p \left( \frac{dv}{dx} - \frac{du}{dx} \right) \bigg|_a^b \).

\[
p \left( \frac{dv}{dx} - \frac{du}{dx} \right) \bigg|_a^b = p(b) \left( u(b) \frac{dv}{dx}(b) - v(b) \frac{du}{dx}(b) \right) - p(a) \left( u(a) \frac{dv}{dx}(a) - v(a) \frac{du}{dx}(a) \right) = u(b)p(b) \frac{dv}{dx}(b) - v(b)p(b) \frac{du}{dx}(b) - \left[ u(b)p(b) \frac{dv}{dx}(b) - v(b)p(b) \frac{du}{dx}(b) \right] = 0
\]

Therefore, the boundary conditions yield a self-adjoint problem.

4. Text exercise 5.5.8, p. 176.

**Solution**

(a) We will show that \( uL(v) - vL(u) \) is an exact differential as follows.

\[
uL(v) = u \frac{d^4v}{dx^4} = \frac{d}{dx} \left( \frac{d^3v}{dx^3} \right) - \frac{du}{dx} \frac{d^3v}{dx^3} = \frac{d}{dx} \left( \frac{d^3v}{dx^3} \right) - \frac{d}{dx} \left( \frac{du}{dx} \frac{d^2v}{dx^2} \right) + \frac{d^2u}{dx^2} \frac{d^2v}{dx^2}
\]

Analogously, \( vL(u) \) is given by

\[
vL(u) = v \frac{d^4u}{dx^4} = \frac{d}{dx} \left( \frac{d^3u}{dx^3} \right) - \frac{dv}{dx} \frac{d^3u}{dx^3} = \frac{d}{dx} \left( \frac{d^3u}{dx^3} \right) - \frac{d}{dx} \left( \frac{dv}{dx} \frac{d^2u}{dx^2} \right) + \frac{d^2v}{dx^2} \frac{d^2u}{dx^2}
\]

It follows that

\[
uL(v) - vL(u) = \frac{d}{dx} \left( \frac{d^3v}{dx^3} - \frac{du}{dx} \frac{d^2v}{dx^2} - \frac{d^3u}{dx^3} + \frac{dv}{dx} \frac{d^2u}{dx^2} \right) \tag{10}
\]
(b) The integral is evaluated as follows.

\[
\int_0^1 [uL(v) - vL(u)] \, dx \\
= \int_0^1 \frac{d}{dx} \left( \frac{d^3 v}{dx^3} - \frac{du}{dx} \frac{d^2 v}{dx^2} - v \frac{d^3 u}{dx^3} + \frac{dv}{dx} \frac{d^2 u}{dx^2} \right) \, dx \\
= \frac{d^3 v}{dx^3} \bigg|_{x=0}^{x=1} - \frac{du}{dx} \frac{d^2 v}{dx^2} \bigg|_{x=0}^{x=1} - v \frac{d^3 u}{dx^3} \bigg|_{x=0}^{x=1} + \frac{dv}{dx} \frac{d^2 u}{dx^2} \bigg|_{x=0}^{x=1} \\
= u(1) \frac{d^3 v}{dx^3}(1) - \frac{du}{dx}(0) \frac{d^2 v}{dx^2}(0) - v(1) \frac{d^3 u}{dx^3}(1) + \frac{dv}{dx}(1) \frac{d^2 u}{dx^2}(1) \\
- u(0) \frac{d^3 v}{dx^3}(0) + \frac{du}{dx}(0) \frac{d^2 v}{dx^2}(0) + v(0) \frac{d^3 u}{dx^3}(0) - \frac{dv}{dx}(0) \frac{d^2 u}{dx^2}(0) \\
= (u(1) - u(0)) \frac{d^3 v}{dx^3}(0) + (\frac{du}{dx}(0) - \frac{du}{dx}(1)) \frac{d^2 v}{dx^2}(0) + (v(1) - v(0)) \frac{d^3 u}{dx^3}(0) + (\frac{dv}{dx}(1) - \frac{dv}{dx}(0)) \frac{d^2 u}{dx^2}(0) \\
= \int_0^1 [uL(v) - vL(u)] \, dx = 0 \\
\]

(c) Using \( u(0) = v(0) = 0, u(1) = v(1) = 0, \frac{du}{dx}(0) = \frac{dv}{dx}(0) = 0 \) in (11), we obtain \( \int_0^1 [uL(v) - vL(u)] \, dx = 0 \).

(d) There are different ways to choose the boundary conditions. For example, the boundary conditions \( \phi''(1) = 0, \phi''(0) = 0, \phi'(1) = 0 \) and \( \phi''(0) = 0 \) imply that \( \int_0^1 [uL(v) - vL(u)] \, dx = 0 \).

(e) Let \( \phi(x) \) and \( \phi_n(x) \) be eigenfunctions of the differential equation with the eigenfunctions satisfying the boundary conditions in part (c). By construction, \( L(\phi_n) = -\lambda_n e^x \phi \) and \( L(\phi_m) = -\lambda_m e^x \phi \). Applying Green’s formula with \( u = \phi_n(x) \) and \( v = \phi_m(x) \), we have

\[
\int_0^1 [\phi_n(x)L(\phi_m) - \phi_m(x)L(\phi_n)] \, dx = \int_0^1 [-\phi_n(x)\lambda_m e^x \phi_m(x) + \phi_m(x)\lambda_n e^x \phi_n(x)] \, dx \\
= \int_0^1 [-\lambda_m e^x \phi_n(x)\phi_m(x) + \lambda_n e^x \phi_n(x)\phi_m(x)] \, dx \\
= (\lambda_n - \lambda_m) \int_0^1 e^x \phi_n(x)\phi_m(x) \, dx = 0 \\
\]

By assumption, \( \lambda_n \neq \lambda_m \). Therefore, the eigenfunctions are orthogonal with weighting function \( e^x \).

5. Text exercise 5.6.1, part (a) only, p. 188.

**Solution** Using Rayleigh quotient, the lowest eigenvalue \( \lambda_1 \) can be upper bounded as follows

\[
\lambda_1 \leq \frac{-pu_T \frac{du_T}{dx} \bigg|_0^1 + \int_0^1 \left[ p \left( \frac{du_T}{dx} \right)^2 - qu_T^2 \right] \, dx}{\int_0^1 u_T^2 \sigma \, dx} \\
\]

where \( u_T \), the trial function, is a continuous function satisfying the boundary conditions. Noting that \( p = 1, q = -x^2 \) and \( \sigma = 1 \) in the Sturm-Liouville form, we have

\[
\lambda_1 \leq \frac{-u_T \frac{du_T}{dx} \bigg|_0^1 + \int_0^1 \left[ \left( \frac{du_T}{dx} \right)^2 + x^2 u_T^2 \right] \, dx}{\int_0^1 u_T^2 \, dx} \\
= \frac{-u_T \frac{du_T}{dx} \bigg|_0^1 + \int_0^1 \left[ \left( \frac{du_T}{dx} \right)^2 + x^2 u_T^2 \right] \, dx}{\int_0^1 u_T^2 \, dx} \\
\]

(12)
Consider a trial function of form \( u_T(x) = ax^2 + bx + c \) with \( a, b \) and \( c \) to be determined. Applying the first boundary condition, \( \frac{du_T}{dx}(0) = b = 0 \). Hence, \( u_T(x) = ax^2 + c \). Applying the second boundary condition, \( u_T(1) = a + c = 0 \). Therefore, \( u_T(x) = a(x^2 - 1) \). Setting \( a = 1 \), consider the trial function \( u_T(x) = x^2 - 1 \). The trial function has no interior zeros in the interval as desired. Using this trial function in (12), we obtain

\[
\lambda_1 \leq \frac{\frac{148}{105} \approx 2.643}{14} \approx 2.643
\]

**Remark:** The upper bound here is not unique. The bound depends on the choice of the trial function. The “closer” the trial function is to the eigenfunction, the sharper the bound.

6. The vertical displacement of a vibrating rectangular membrane satisfies

\[
u_{tt} = c^2(u_{xx} + u_{yy}), \quad 0 \leq x \leq L, \quad 0 \leq y \leq H, \quad t \geq 0
\]

with boundary conditions

\[
u(0, y, t) = u(L, y, t) = u_y(x, 0, t) = u_y(x, H, t) = 0
\]

and initial conditions

\[
u(x, y, 0) = \alpha(x, y), \quad u_t(x, y, 0) = 0
\]

Use separation of variables to find the solution \( u(x, y, t) \), i.e. start by looking for solutions of the form \( u(x, y, t) = \Phi(x, y)T(t) \). Determine the natural frequencies of vibration, and sketch contours of some of the eigenmodes as was done in class for a similar problem. (Note that some of the boundary conditions are of Neumann type.)

**Solution** Using the method of separation of variables, assume \( u(x, t) = \phi(x, y)T(t) \). Substituting this product form in the PDE, we obtain

\[
\phi(x)T''(t) = c^2[T(t)\phi_{xx} + T(t)\phi_{yy}]
\]

Dividing both sides of the equation above by \( c^2\phi(x, y)T(t) \) results

\[
\frac{T''}{c^2T} = \frac{\phi_{xx} + \phi_{yy}}{\phi} = -\lambda
\]

A function of time will be equal to a function of space if both functions are equal to the same constant. Therefore,

\[
\frac{T''}{c^2T} = \frac{\phi_{xx} + \phi_{yy}}{\phi} = -\lambda \quad (13)
\]

where \( \lambda \) is some constant. (13) results an ODE for \( h(t) \) and a PDE for \( \phi(x, y) \). The PDE resulting from (13) is given by \( \phi_{xx} + \phi_{yy} = -\lambda \phi \). We now consider the boundary conditions. The first boundary condition \( u(0, y, t) = 0 \) implies that \( \phi(0, y)T(t) = 0 \). For nontrivial solutions, set \( \phi(0, y) = 0 \). In an analogous manner, \( \phi(L, y) = 0, \phi_y(x, 0) = 0 \) and \( \phi_y(x, H) = 0 \). With this, consider the following problem.

\[
\phi_{xx} + \phi_{yy} = -\lambda \phi \quad (14)
\]

\[
\phi(0, y) = 0 \quad (15)
\]

\[
\phi(L, y) = 0 \quad (16)
\]

\[
\phi_y(x, 0) = 0 \quad (17)
\]

\[
\phi_y(x, H) = 0 \quad (18)
\]
To solve (14), assume a solution \( \phi(x, y) = f(x)g(y) \) using separation of variables. Substituting this product form in the PDE, we obtain
\[
g(y)f''(x) + f(x)g''(y) = -\lambda f(x)g(y)
\]
Dividing both sides of the equation above by \( f(x)g(y) \) results
\[
\frac{f''(x)}{f} = -\lambda - \frac{g''(y)}{g}
\]
A function of \( x \) will be equal to a function of \( y \) if both functions are equal to the same constant. Therefore,
\[
\frac{f''(x)}{f} = -\lambda - \frac{g''(y)}{g} = -\mu \quad (19)
\]
where \( \mu \) is some constant. The first ODE resulting from (19) is given by \( f''(x) = -\mu f \).
Using the boundary condition in (15), \( \phi(0, y) = f(0)g(y) = 0 \). For nontrivial solutions, set \( f(0) = 0 \). In an analogous manner, using the boundary condition in (16), we obtain \( f(L) = 0 \). With this, consider the following problem.
\[
f''(x) = -\mu \quad (20)
f(0) = 0 \quad (21)
f(L) = 0 \quad (22)
\]
The differential equation in (20) with boundary conditions (21) and (22) has been considered before. Recall that the eigenvalues and eigenfunctions are respectively given by
\[
\mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, ...
\]
\[
f_n(x) = \sin \frac{n\pi x}{L} \quad (24)
\]
Next, consider the second ODE resulting from (19) which is given by \( g''(y) = -(\lambda - \mu)g \).
Using the boundary condition in (17), \( \phi_y(x, 0) = f(0)g'(0) = 0 \). For nontrivial solutions, set \( g'(0) = 0 \). In an analogous manner, using the boundary condition in (18), we obtain \( g'(H) = 0 \). With this, consider the following problem.
\[
g''(y) = -(\lambda - \mu)g \quad (25)
g'(0) = 0 \quad (26)
g'(H) = 0 \quad (27)
\]
The differential equation in (25) with boundary conditions (26) and (27) has been considered before. For a given \( \mu_n \), recall that the eigenvalues and eigenfunctions are respectively given by
\[
\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m = 0, 1, 2, 3, ...
\]
\[
g_{nm}(y) = \cos \frac{m\pi y}{H} \quad (29)
\]
where the \( m = 0 \) case results from considering the case \( \lambda = \mu \) in (25) and obtaining an eigenfunction of 1. Using (23) and (28), \( \lambda_{nm} \) has the following representation.
\[
\lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2
\]
(30)
The two-dimensional eigenfunction corresponding to $\lambda_{nm}$ is given by

$$\phi_{nm}(x, y) = \sin \frac{n\pi x}{L} \cos \frac{m\pi y}{H}, \quad n = 1, 2, 3, \ldots \quad m = 0, 1, 2, 3, \ldots \quad (31)$$

This completes the analysis of the PDE for $\phi(x, y)$. We next consider the ODE for $h(t)$ that results from (13). The ODE is given by

$$T''(t) = -\lambda c^2 T(t)$$

The general solution is given by

$$T(t) = A_{nm} \cos c\sqrt{\lambda_{nm}} t + B_{nm} \sin c\sqrt{\lambda_{nm}} t \quad (32)$$

where $A_{nm}$ are $B_{nm}$ are some constants. Using the principle of superposition, the solution to the PDE can be written as follows.

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[ A_{nm} \sin \frac{n\pi x}{L} \cos \frac{m\pi y}{H} \cos c\sqrt{\lambda_{nm}} t + B_{nm} \sin \frac{n\pi x}{L} \cos \frac{m\pi y}{H} \sin c\sqrt{\lambda_{nm}} t \right] \quad (33)$$

To determine $A_{nm}$ and $B_{nm}$, we consider the initial conditions. Applying the first initial condition, we have

$$u(x, y, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin \frac{n\pi x}{L} \cos \frac{m\pi y}{H} = \alpha(x, y)$$

The above equation implies that

$$A_{nm} = \frac{4}{LH} \int_0^L \int_0^H \alpha(x, y) \sin \frac{n\pi x}{L} \cos \frac{m\pi y}{H} \, dy \, dx \quad (34)$$

where the result follows from the orthogonality of the eigenfunctions $\phi_{nm}$ in (31). Applying the second initial condition, we have

$$u_t(x, y, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c\sqrt{\lambda_{nm}} B_{nm} \sin \frac{n\pi x}{L} \cos \frac{m\pi y}{H} = 0$$

By the orthogonality of the eigenfunctions $\phi_{nm}$ and since $c\sqrt{\lambda_{nm}} \neq 0$, the above equation implies that $B_{nm} = 0$. With this, using (33), the solution to the PDE has the following form.

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin \frac{n\pi x}{L} \cos \frac{m\pi y}{H} \cos c\sqrt{\lambda_{nm}} t \quad (35)$$

with $A_{nm}$ determined from (34). The natural frequencies of vibration are given by

$$c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$

Figures 1-5 below show contours of some of the eigenmodes.
Figure 1: Modes of a vibrating rectangular membrane. $n=1, m=0$

Figure 2: Modes of a vibrating rectangular membrane. $n=1, m=1$

Figure 3: Modes of a vibrating rectangular membrane. $n=1, m=2$
Figure 4: Modes of a vibrating rectangular membrane. n=2, m =1

Figure 5: Modes of a vibrating rectangular membrane. n=2, m =3