Example. The vertical displacement $u(x, y, t)$ of a rectangular membrane satisfies

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad 0 < x < L, \quad 0 < y < H, \quad t > 0$$

with initial conditions $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = g(x, y)$, and with homogeneous Dirichlet boundary conditions $u = 0$ on all four sides. The length $L$, width $H$, wave speed $c$, and functions $f(x, y)$ and $g(x, y)$ are considered to be known.

(a) Look for separable solutions of the form $u(x, y, t) = \Phi(x, y)T(t)$, and solve the eigenvalue problem for $\Phi(x, y)$.

(b) Use superposition to determine the general solution satisfying the PDE and homogeneous boundary conditions, and then use the initial conditions to obtain the solution.

(c) Consider the case with $g(x, y) = 0$. Describe the modes of vibration of the membrane.