Example. The temperature $u(x, t)$ in a bar of length $L$ satisfies the heat flow problem

$$u_t = ku_{xx}, \quad 0 < x < L, \quad t > 0$$

with initial condition $u(x, 0) = f(x)$ and Dirichlet boundary conditions $u(0, t) = u(L, t) = 0$. The length $L$, diffusitivity $k$ and function $f(x)$ are considered to be known.

(a) Use separation of variables to determine separated equations (ODEs). What initial conditions and/or boundary conditions can be applied to these equations?

(b) Identify the eigenvalue problem and find all non-trivial solutions.

(c) Use superposition to determine the general solution satisfying the PDE and homogeneous boundary conditions.

(d) Determine the solution for the case

$$f(x) = 2 \sin \left(\frac{\pi x}{L}\right) - 5 \sin \left(\frac{3\pi x}{L}\right)$$