Example. The temperature $u(x, y)$ in a 2D rectangular slab satisfies the steady heat flow problem

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b$$

with Dirichlet boundary conditions $u(0, y) = u(a, y) = u(x, 0) = 0$ and $u(x, b) = f(x)$. The length $a$, width $b$ and function $f(x)$ are considered to be known.

(a) Use separation of variables to determine separated equations (ODEs). What boundary conditions can be applied to these equations?

(b) Identify the eigenvalue problem and find all non-trivial solutions.

(c) Use superposition to determine the general solution satisfying the PDE and homogeneous boundary conditions.

(d) Determine the solution for the case

$$f(x) = \begin{cases} 
  x & \text{if } 0 \leq x < a/2 \\
  0 & \text{if } a/2 \leq x \leq a
\end{cases}$$