Example. Steady state temperature $u(r, \theta, z)$ in a half-cylinder satisfies

$$\frac{1}{r} \left( ru_r \right)_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0, \quad 0 < r < a, \quad 0 < \theta < \pi, \quad 0 < z < H$$

with homogeneous boundary conditions $u = 0$ on faces of the cylinder with $\theta = 0$ and $\theta = \pi$, and on the curved face with $r = a$. The boundary conditions on the top face of the cylinder at $z = H$ is $u = 0$, while the bottom face at $z = 0$ is $u(r, \theta, 0) = V(r, \theta)$.

(a) Look for separable solutions of the form $u(r, \theta, z) = f(r)g(\theta)h(z)$, and identify all eigenvalue problems in the $r$, $\theta$ and/or $z$ directions.

(b) Solve the eigenvalue problems and use superposition to determine the general solution satisfying the PDE and homogeneous boundary conditions.

(c) Find the solution by applying the nonhomogeneous boundary condition at $z = 0$. 