1. Consider the system
\[ \mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x} \]

Find the general solution of the system and express this solution in terms of real-valued functions. Describe the behavior of solution trajectories in the \( x_1 - x_2 \) phase plane. Is the critical point at \( \mathbf{x} = 0 \) stable, asymptotically stable, or unstable?

First, assume the solution is of the form \( \mathbf{x} = \mathbf{z} e^{rt} \) and consider the determinant condition
\[
\det \begin{bmatrix} 1 - r & -1 \\ 5 & -3 - r \end{bmatrix} = 0
\]
\[
(1 - r)(-3 - r) + 5 = 0
\]
\[
r^2 + 2r + 2 = 0
\]
which leads to \( r = -1 \pm i \). Consider the first eigenvalue, \( r_1 = -1 + i \), to find the corresponding eigenvector \( \mathbf{z}^{(1)} \)
\[
\begin{bmatrix} 2 - i & -1 \\ 5 & -2 - i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
Taking the first equation, we have \( z_2 = (2 - i)z_1 \), so \( \mathbf{z} = \begin{bmatrix} 1 \\ 2 - i \end{bmatrix} \). Using the eigenvector just found, and the corresponding eigenvalue, we can now find the solution.
\[
\begin{bmatrix} 1 \\ 2 - i \end{bmatrix} e^{(-1+i)t} = e^{-t} \begin{bmatrix} 1 \\ 2 - i \end{bmatrix} e^{it}
\]
\[
= e^{-t} \begin{bmatrix} 1 \\ 2 - i \end{bmatrix} (\cos(t) + i \sin(t))
\]
\[
= e^{-t} \begin{bmatrix} \cos(t) \\ 2 \cos(t) + \sin(t) \end{bmatrix} + ie^{-t} \begin{bmatrix} \sin(t) \\ -\cos(t) + 2 \sin(t) \end{bmatrix}
\]
so,
\[
\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} \cos(t) \\ 2 \cos(t) + \sin(t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin(t) \\ -\cos(t) + 2 \sin(t) \end{bmatrix}
\]

Now, since we have a solution, we could specify an initial condition and then plot points. Taking
\[
\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
Leads to
\[
\mathbf{x}(t) = e^{-t} \begin{bmatrix} \cos(t) + 2 \sin(t) \\ 3 \sin(t) \end{bmatrix}
\]
\[
\mathbf{x}(\pi/2) = \begin{bmatrix} 2e^{-\pi/2} \\ 3e^{-\pi/2} \end{bmatrix}
\]
\[
\mathbf{x}(\pi) = \begin{bmatrix} -e^{-\pi} \\ 0 \end{bmatrix}
\]
Based on the path the points are following, the spiral is rotating counterclockwise as \( t \) increases. This is shown below in Figure 1.
Figure 1: Phase portrait for problem 1. The plot shows a stable spiral around the origin. Black line shows the solution for the picked initial condition \([x_1(0) = 1, x_2(0) = 0]\).

2. Consider the nonlinear system

\[
\frac{dx}{dt} = 3 - 3y^2, \quad \frac{dy}{dt} = 6 + 3x
\]

Find all critical points of the nonlinear system and determine the linearized equations about each critical point. Examine the local behavior about each critical point and use this information to sketch the phase portrait for the system. (You might use Maple to confirm your sketch.)

First, to find the critical points, set both derivatives to zero.

\[
0 = 3 - 3y^2, \quad 0 = 6 + 3x
\]

The first equation gives \(y = \pm 1\) and the second gives \(x = -2\). So, the critical points are \((-2, 1)\) and \((-2, -1)\).

Now, find the Jacobian matrix for the system, with \(\frac{dx}{dt} = f(x, y)\) and \(\frac{dy}{dt} = g(x, y)\)

\[
J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & -6y \\ 3 & 0 \end{bmatrix}
\]

Near \((-2, 1)\) the linearized system is

\[
\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

which has the eigenvalues \(r_1 = i3\sqrt{2}, r_2 = -i3\sqrt{2}\). Based on the eigenvalues, the local behavior around \((-2, 1)\) will be a center in the phase portrait. The only piece of information
missing is the direction of rotation. To find this, calculate the derivative at some initial condition, say \[ \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \]

\[
\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} .
\]

Hence, the derivative points upward from \((u(0), v(0)) = (1, 0)\), so the trajectories in the phase portrait will be going counterclockwise around \((-2, 1)\).

Near \((-2, -1)\) the linearized system is

\[
\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

which has the eigenvalues \(r_1 = 3\sqrt{2}, r_2 = -3\sqrt{2}\) and the corresponding eigenvectors \(z^{(1)} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, z^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}\). Based on the eigenvalues, we have a saddle near \((-2, -1)\), where the solution goes toward \((-2, -1)\) along the second eigenvector and away from \((-2, -1)\) along the first eigenvector.

Putting these two behaviors together, we have the phase portrait shown in Figure 2.

Figure 2: Phase portrait for problem 2. Phase portrait displays a center near \((-2, 1)\) and a saddle near \((-2, -1)\).
3. Populations of two competing species are given by \( x(t) \) and \( y(t) \). These populations are governed by

\[
\frac{dx}{dt} = x(3 - 2x - y), \quad \frac{dy}{dt} = y(4 - y - 3x).
\]

Find all critical points of the system and determine the local behavior of the solution about each critical point. Sketch the behavior of solutions in the phase plane for \( x \geq 0 \) and \( y \geq 0 \). Is a stable coexistence of the two populations possible? Explain.

Start by finding the critical points. Setting \( \frac{dx}{dt} \) to zero, we get

\[
x = 0 \quad \text{or} \quad x = \frac{3 - y}{2}.
\]

From \( x = 0 \), setting \( \frac{dy}{dt} = 0 \) gives \( y = 0 \) or \( y = 4 \).

From \( x = \frac{3 - y}{2} \), setting \( \frac{dy}{dt} \) gives \( y = 0 \) or \( y = 1 \).

So, the critical points are \((0, 0), (0, 4), \left(\frac{3}{2}, 0\right), (1, 1)\).

Now, the Jacobian matrix of the system is

\[
J(x, y) = \begin{bmatrix}
3 - 4x - y & -x \\
-3y & 4 - 2y - 3x
\end{bmatrix}
\]

Near \((0, 0)\), the linearized system is

\[
\begin{bmatrix}
u' \\
v'
\end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

which has the eigenvalues \( r_1 = 3, r_2 = 4 \) and the corresponding eigenvectors \( z^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( z^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). So, the behavior near \((0, 0)\) will be an unstable node.

Near \((0, 4)\), the linearized system is

\[
\begin{bmatrix}
u' \\
v'
\end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -12 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

which has the eigenvalues \( r_1 = -1, r_2 = -4 \) and the corresponding eigenvectors \( z^{(1)} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} \), \( z^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). So, the behavior near \((0, 4)\) will be a stable node.

Near \(\left(\frac{3}{2}, 0\right)\), the linearized system is

\[
\begin{bmatrix}
u' \\
v'
\end{bmatrix} = \begin{bmatrix} -3 & -\frac{3}{2} \\ 0 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

which has the eigenvalues \( r_1 = -3, r_2 = -\frac{3}{2} \), with the corresponding the eigenvectors \( z^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( z^{(2)} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix} \). So, the behavior near \(\left(\frac{3}{2}, 0\right)\) will be a stable node.

Near \((1, 1)\), the linearized system is

\[
\begin{bmatrix}
u' \\
v'
\end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]
which has the eigenvalues $r = \frac{-3 \pm \sqrt{13}}{2}$ with the corresponding eigenvectors $z^{(1)} = \begin{bmatrix} -\frac{1}{3}(1 - \sqrt{13}) \\ 1 \end{bmatrix}$, $z^{(2)} = \begin{bmatrix} \frac{1}{6}(1 + \sqrt{13}) \\ 1 \end{bmatrix}$. So, the behavior near $(1, 1)$ will be a saddle.

Since the only critical point where both $x > 0$ and $y > 0$ is unstable, coexistence is not possible.

Figure 3: Phase portrait for problem 3. Phase portrait displays an unstable node near $(0, 0)$, a stable node near $(\frac{3}{2}, 0)$, a stable node near $(0, 4)$, and a saddle near $(1, 1)$. 