1. Let

\[ A = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 1 & 1 \\ -4 & -2 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \]

Show that the column vectors of \( A \) are linearly independent, and use row reduction followed by backwards substitution to find a vector \( x \) such that \( Ax = b \).

To check that the column vectors of \( A \) are linearly independent, consider

\[ c_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0 \]

This gives the system

\[
\begin{bmatrix}
2 & 3 & 2 \\
4 & 1 & 1 \\
-4 & -2 & -1
\end{bmatrix} \begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

So, if we create an augmented system and perform row operations, we can solve for \( c_1, c_2, \) and \( c_3 \).

Applying row operations (listed to the right of the resulting matrix) to the augmented system to get an upper triangular system gives

\[
\begin{bmatrix}
2 & 3 & 2 & 1 \\
0 & -5 & -3 & 0 \\
0 & 4 & 3 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 3 & 2 & 0 \\
0 & -5 & -3 & 0 \\
0 & 0 & \frac{3}{5} & 0
\end{bmatrix}
\]

which can be solved using back substitution to find \( c_1, c_2, c_3 = 0 \). Since each of \( c_1, c_2, c_3 \) are zero, we have that the columns of \( A \) are linearly independent.

Alternatively, one can show that the columns of a matrix are linearly independent by showing that the determinant of the matrix is not zero.

\[
\det \begin{bmatrix} 2 & 3 & 2 \\ 4 & 1 & 1 \\ -4 & -2 & -1 \end{bmatrix} = 2(-1 + 2) - 3(-4 + 4) + 2(-8 + 4) = -6 \neq 0
\]

Since the determinant is not zero, the columns of \( A \) must be linearly independent.

Using the same row operations as above and then back substitution, the system \( Ax = b \) can be solved.

\[
\begin{bmatrix}
2 & 3 & 2 & 1 \\
0 & -5 & -3 & 2 \\
0 & 0 & \frac{3}{5} & \frac{3}{5}
\end{bmatrix} \rightarrow R_3 + \frac{4}{5}R_2 \rightarrow R_3
\]
So, the three equations we have are

\[
\begin{align*}
2c_1 + 3c_2 + 2c_3 &= 1 \\
-5c_2 - 3c_3 &= 2 \\
\frac{3}{5}c_3 &= \frac{3}{5}
\end{align*}
\]

Solving this with back substitution yields \( c_3 = 1, c_2 = -1, c_3 = 1 \).

2. Consider the constant-coefficient system

\[
x' = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix} x
\]

Find the general solution and describe its behavior in the phase plane.

First, assume the solution has the form \( x = ze^{rt} \). To find \( r \), use the determinant condition

\[
\det \left( \begin{bmatrix} 4 - r & -3 \\ 6 & -5 - r \end{bmatrix} \right) = 0
\]

\[
(4 - r)(-5 - r) + 18 = 0
\]

\[
r^2 + r - 2 = 0
\]

This gives \( r_1 = 1 \) and \( r_2 = -2 \). Using each \( r \), we will now find a corresponding \( z \). For \( r_1 = 1 \), the system is

\[
\begin{bmatrix} 3 & -3 \\ 6 & -6 \end{bmatrix} z^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Using the first equation, we have \( z_1^{(1)} = z_2^{(1)} \), so \( z^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

For \( r_2 = -2 \), the system is

\[
\begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix} z^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Using the first equation, we have \( 2z_1^{(2)} = z_2^{(2)} \), so \( z^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

Combining everything together, the solution is then

\[
x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}
\]

3. Solve the initial-value problem

\[
x' = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

Describe the behavior of the solution in the phase plane as \( t \to \infty \).
Assuming the solution has the form \( x = z e^{rt} \), first find \( r \) by considering the determinant condition for \( r \)

\[
\det \left( \begin{bmatrix} 1 - r & 1 \\ -3 & 5 - r \end{bmatrix} \right) = 0
\]

\[
(1 - r)(5 - r) + 3 = 0
\]

\[
r^2 - 6r + 8 = 0
\]

This gives \( r_1 = 2 \) and \( r_2 = 4 \). For \( r_1 = 2 \), the system for \( z^{(1)} \) is

\[
\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} z^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

From the first equation, \(-z_1^{(1)} + z_2^{(1)} = 0\), so \( z^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

Moving to the second \( r \), \( r_2 = 4 \) yields the system

\[
\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} z^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Using the first equation, \(-3z_1^{(2)} + z_2^{(2)} = 0\), so \( z^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \).

With \( r_1, r_2, z^{(1)} \), and \( z^{(2)} \) gives the solution

\[
x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t}
\]

Applying the initial conditions leads to

\[
x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]

Solving this system for \( c_1 \) and \( c_2 \) gives \( c_1 = -1, c_2 = 1 \).

\[
x = \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t}
\]
Figure 2: Phase portrait for problem 3. The black line represents the solution for the initial condition $x_1(0) = 0$ and $x_2(0) = 2$. ($x = x_1, y = x_2$ on the graph)