1. Consider the nonhomogeneous equation

\[ t^2 y'' + ty' - 4y = 4 - t^2, \quad t > 0 \]

(a) Show that \( y_1(t) = t^{-2} \) and \( y_2(t) = t^2 \) are independent solutions of the corresponding homogeneous equation.

For \( y_1 \) and \( y_2 \) to be independent solutions, both must satisfy the homogeneous ODE

\[ t^2 y'' + ty' - 4y = 0 \]

and the Wronskian must not be zero for \( t > 0 \).

First, to show that \( y_1 \) satisfies the ODE

\[ y_1' = -2t^{-3}, \quad y_1'' = 6t^{-4} \]

so,

\[ t^2(6t^{-4}) + t(-2t^{-3}) - 4(t^{-2}) = 6t^{-2} - 2t^{-2} + 4t^{-2} = 0 \]

Now, see if \( y_2 \) satisfies the ODE

\[ y_2' = 2t, \quad y_2'' = 2 \]

so,

\[ t^2(2) + t(2t) - 4(t^2) = 0 \]

For the Wronskian,

\[ W = \det \begin{bmatrix} t^{-2} & t^2 \\ -2t^{-3} & 2t \end{bmatrix} = 2t^{-1} + 2t^{-1} \neq 0, \quad t > 0 \]

Thus, \( y_1 \) and \( y_2 \) are independent solutions of the homogeneous ODE.

(b) Use variation of parameters to find the general solution of the nonhomogeneous equation.

First, get the ODE in standard form

\[ y'' + t^{-1}y' - 4t^{-2}y = 4t^{-2} - 1 \]

For variation of parameters, the solution form is

\[ y = u_1(t)y_1(t) + u_2(t)y_2(t) \]

\[ u_1' = -\frac{y_2g(t)}{W} = -\frac{t^2(4t^{-2} - 1)}{4t^{-1}} = -t + t^3/4 \quad \to \quad u_1 = -t^2/2 + t^4/16 + c_1 \]

\[ u_2' = \frac{y_1g(t)}{W} = \frac{t^{-2}(4t^{-2} - 1)}{4t^{-1}} = t^{-3} - t^{-1}/4 \quad \to \quad u_2 = -t^{-2}/2 - \ln(t)/4 + c_2 \]

The solution is then

\[ y = (-t^2/2 + t^4/16 + c_1)t^{-2} + (-t^{-2}/2 - \ln(t)/4 + c_2)t^2 \]
2. The displacement $u(t)$ of a forced mass-spring-damper system satisfies

$$2u'' + \gamma u' + 8u = 4 \sin(\omega t), \quad u(0) = u'(0) = 0$$

where $\gamma \geq 0$ is a damping coefficient and $\omega > 0$ is the frequency of the external forcing. (a) Find all values of $\gamma$ and $\omega$ for which resonance occurs. Find $u(t)$ for this case.

For resonance to occur there are two things necessary. There has to be no damping and the forcing frequency has to be the same as the natural frequency ($\omega_0$) of the system. Thus, $\gamma = 0$ and to find the natural frequency, we need the ODE to be in standard form:

$$u'' + 4u = 2 \sin(\omega t)$$

Solving the homogeneous problem, $u_H = c_1 \cos(2t) + c_2 \sin(2t)$. So, $\omega_0 = \sqrt{4} = 2$. Using the method of undetermined coefficients, the particular solution can be found. The guess is:

$$u_p = t(A \sin(2t) + B \cos(2t))$$

Now, to find $A$ and $B$.

$$u'_p = t(2A \cos(2t) - 2B \sin(2t)) + A \sin(2t) + B \cos(2t)$$

$$u''_p = 4A \cos(2t) - 4B \sin(2t) + t(-4A \sin(2t) - 4B \cos(2t))$$

Plugging into the ODE and matching coefficients, we have

$$-4B = 2 \quad (\sin(2t))$$

$$4A = 0 \quad (\cos(2t))$$

$$-4A + 4A = 0 \quad (t \sin(2t))$$

$$-4B + 4B = 0 \quad (t \cos(2t))$$

Giving $A = 0$ and $B = -1/2$. Thus, the general solution is

$$u(t) = c_1 \cos(2t) + c_2 \sin(2t) - 1/2t \cos(2t)$$

Applying the initial conditions

$$u(0) = c_1 = 0$$

$$u'(0) = 2c_2 - 1/2 = 0 \quad \rightarrow \quad c_2 = 1/4$$

So, the solution is

$$u(t) = \frac{1}{4} \sin(2t) - \frac{1}{2}t \cos(2t)$$

(b) Find the amplitude of the forced response when $\gamma = 1$ and $\omega = 1$. With $\gamma$ and $\omega$ given, we have the ODE

$$2u'' + u' + 8u = 4 \sin(t)$$

Since the forced response is the focus, we only need to find the particular solution. Due to the presence of damping, the homogeneous solutions and particular solution can’t coincide (we can’t have resonance), so we can jump to guessing the particular solution without knowing the homogeneous solution using the method of undetermined coefficients.

$$u_p = A \sin(t) + B \cos(t)$$
To find \( A \) and \( B \), we plug the guess into the ODE

\[
\begin{align*}
    u' &= A \cos(t) - B \sin(t) \\
    u'' &= -A \sin(t) - B \cos(t)
\end{align*}
\]

Matching coefficients

\[
\begin{align*}
    -2A - B + 8A &= 4 \quad (\sin(t)) \\
    -2B + A + 8B &= 0 \quad (\cos(t))
\end{align*}
\]

So, \( A = 24/37 \) and \( B = -4/37 \) and the amplitude of the forced response is

\[
R = \sqrt{\left(\frac{24}{37}\right)^2 + \left(-\frac{4}{37}\right)^2} = \frac{4\sqrt{37}}{37}
\]

3. Solve the boundary-value problem for \( y(x) \), or else show that no solutions exist.

(a) \( y'' - 2y' - 8y = 0, \quad 0 < x < 2, \quad y(0) = 1, \quad y(2) = 0 \)

Characteristic polynomial is \( r^2 - 2r - 8 = (r - 4)(r + 2) = 0 \), so \( r = 4, -2 \) and the general solution is

\[
y = c_1 e^{4x} + c_2 e^{-2x}
\]

Applying the initial conditions

\[
y(0) = c_1 + c_2 = 1 \\
y(2) = c_1 e^8 + c_2 e^{-4} = 0 \quad \rightarrow \quad c_2 = -e^{12} c_1
\]

Combining the two equations

\[
c_1 = \frac{1}{1 - e^{12}} \\
c_2 = -\frac{e^{12}}{1 - e^{12}}
\]

so, the solution is

\[
y = \frac{1}{1 - e^{12}} \left( e^{4x} - e^{12}e^{-2x} \right)
\]

(b) \( 2y'' + 8y = 3x + e^{-x}, \quad 0 < x < \pi, \quad y(0) = 0, \quad y(\pi) = 0 \)

First, solve the homogeneous problem.

\[
y_H = c_1 \sin(2x) + c_2 \cos(2x)
\]

Guess for the method of undetermined coefficients is

\[
y_p = Ax + B + Ce^{-x}
\]

\[
y'_p = A - Ce^{-x}
\]

\[
y''_p = Ce^{-x}
\]

Matching coefficients

\[
8B = 0 \quad \text{(constant)}
\]
\[ 8A = 3 \quad (x) \]
\[ 2C + 8C = 1 \quad (e^{-x}) \]
so, \( A = \frac{3}{8}, \) \( B = 0, \) and \( C = \frac{1}{10}, \) giving the general solution
\[ y = c_1 \sin(2x) + c_2 \cos(2x) + \frac{3}{8}x + \frac{1}{10}e^{-x} \]

Now that we have the general solution, initial conditions can be applied
\[ y(0) = c_2 + \frac{1}{10} = 0 \quad \rightarrow \quad c_2 = -\frac{1}{10} \]
\[ y(\pi) = -c_2 + \frac{3\pi}{8} + \frac{1}{10}e^{-\pi} = 0 \quad \rightarrow \quad c_2 = \frac{3\pi}{8} + \frac{1}{10}e^{-\pi} \]

From the two initial conditions, we have an inconsistency, so a solution does not exist.