1. Consider the population model given by

\[ \frac{dy}{dt} = -y \left(1 - \frac{y}{T}\right) \left(1 - \frac{y}{K}\right) \]

where \( t \) is time, \( y(t) \) measures the population, \( y_0 \) is the initial population, and \( T \) and \( K \) are positive constants with \( T < K \).

a) Sketch the graph of \( F(y) \) versus \( y \). Find all critical points and determine whether each critical point is asymptotically stable or unstable.

From the graph, \( y_c = 0 \) is asymptotically stable, \( y_c = T \) is unstable, and \( y_c = K \) is asymptotically stable.

b) It is known that the population approaches \( y = 5 \) as \( t \to \infty \) if \( y_0 = 5 \), and that the rate of change of the population is \(-2/5\) when \( y = 1 \). Determine the behavior of \( y(t) \) as \( t \to \infty \) if \( y_0 = 3 \).

From the information provided, we have that \( F(5) = 0 \), so \( y_c = 5 \) is stable, and thus \( K = 5 \). We also have that \( F(1) = -2/5 \). This leads to the equation

\[ \frac{-2}{5} = -\frac{4}{5} \left(1 - \frac{1}{T}\right) \]

which leads to \( T = 2 \).

Since we now know \( T \) and \( K \), we can look at what happens for \( y_0 = 3 \). Since \( T < 3 < K \), \( y(t) \) should approach \( y = 5 \) as \( t \to \infty \).

2. Consider the second-order constant-coefficient equation

\[ 2y'' - y' - 3y = 0 \]

(a) Find the general solution of the differential equation by considering solutions of the form \( y(t) = e^{rt} \), where \( r \) is a constant.

With the solution of the form \( e^{rt} \), take the first two derivatives

\[ y' = re^{rt}; \quad y'' = r^2e^{rt} \]
and plug into the ODE to get

\[ e^r(2r^2 - r - 3) = 0 \]

\[ 2r^2 - r - 3 = (2r - 3)(r + 1) = 0 \]

So, \( r_1 = 3/2; r_2 = -1 \) and the two solutions are \( y_1 = e^{3t/2} \) and \( y_2 = e^{-t} \), which give the general solution

\[ y = c_1 e^{3t/2} + c_2 e^{-t} \]

(b) Find the unique solution satisfying the differential equation and the initial conditions \( y(0) = 3 \) and \( y'(0) = 2 \).

Applying the initial conditions gives the system

\[ 3 = c_1 + c_2 \]

\[ 2 = \frac{3}{2}c_1 - c_2 \]

Solving this gives \( c_1 = 2 \) and \( c_2 = 1 \), so the unique solution is then

\[ y = 2e^{3t/2} + e^{-t} \]

3. Consider a second-order linear equation

\[ t^2 y'' + ty' + 4y = 0, \quad t > 0 \]

(a) Find a constant \( \alpha \) such that \( y_1(t) = \cos(\alpha \ln t) \) and \( y_2(t) = \sin(\alpha \ln t) \) are solutions of the equation.

\[
\begin{align*}
y_1' &= -\sin(\alpha \ln t) \frac{\alpha}{t}; \\
y_1'' &= -\cos(\alpha \ln t) \frac{\alpha^2}{t^2} + \sin(\alpha \ln t) \frac{\alpha}{t^2} \\
y_2' &= \cos(\alpha \ln t) \frac{\alpha}{t}; \\
y_2'' &= -\sin(\alpha \ln t) \frac{\alpha^2}{t^2} - \cos(\alpha \ln t) \frac{\alpha}{t^2}
\end{align*}
\]

plugging in, both \( y_1 \) and \( y_2 \) require \( \alpha^2 - 4 = 0 \), so we can take \( \alpha = 2 \), since \( \alpha = -2 \) is redundant, and \( y_1 \) \( y_2 \) will be solutions.

(b) Compute the Wronskian, \( W(y_1, y_2) \), with the value of \( \alpha \) found in part (a) to determine whether \( y_1(t) \) and \( y_2(t) \) are independent solutions.

\[
W(y_1, y_2) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} \sin(2 \ln t) & \cos(2 \ln t) \\ \cos(2 \ln t) \frac{2}{t} & -\sin(2 \ln t) \frac{2}{t} \end{bmatrix} = \left( -\sin^2(2 \ln t) - \cos^2(2 \ln t) \right) \frac{2}{t} = -\frac{2}{t} \neq 0
\]

So, since the Wronskian is never 0 for any \( t > 0 \), \( y_1 \) and \( y_2 \) are independent solutions.