1. (a) Solve

\[ t^3 y' + 4t^2 y = e^t, \quad y(1) = 0, \quad t > 0 \]

\[ \text{Rewrite} \rightarrow \]
\[ y' + \frac{4}{t} y = \frac{e^t}{t^2} \]

\[ \mu = e^{\int \frac{4}{t} \, dt} = e^{4 \ln t} = t^4 \]

\[ \frac{d}{dt} (yt^4) = te^t \]

\[ yt^4 = \int te^t \, dt \]

\[ u = t \quad dv = e^t \, dt \]
\[ du = dt \quad v = e^t \]

\[ = te^t - e^t + C \]

\[ y = \frac{1}{t^3} te^t - \frac{e^t}{t^4} + \frac{C}{t^4} \]
\[ y(1) = e^0 e + c = 0 \]

\[ c = e^0 = 1 \]

\[ y(t) = \frac{e^t}{t^3} - \frac{e^t}{t^4} \]
1. (b) Solve the initial-value problem

\[
\frac{ty'}{t^2 + 1} + y^3 = 0, \quad y(1) = -\frac{1}{2}
\]

Write your solution in explicit form, i.e. give $y$ explicitly as a function of $t$.

Rewrite →

\[
\frac{dy}{dt} = -y^3 \left(\frac{t^2 + 1}{t}\right)
\]

\[
-\int \frac{dy}{y^3} = \int \left(t + \frac{1}{t}\right) dt
\]

\[
\frac{1}{2y^2} = \frac{1}{2} t^2 + \ln t + C
\]

\[
y^2 = \frac{1}{t^2 + 2\ln t + C}
\]

Apply I.C. →

\[
\frac{1}{4} = \frac{1}{1 + C} \quad \Rightarrow \quad C = 3
\]

\[
y = \pm \sqrt{\frac{1}{t^2 + 2\ln t + 3}}
\]
Choose minus sign to satisfy
I.e:

\[ y = - \sqrt{\frac{1}{t^2 + 2\ln t + 3}} \]
2. A certain population \( y(t) \) of insects is well-modeled by a logistic equation of the form

\[
\frac{dy}{dt} = (r - ay)y, \quad \text{where } r \text{ and } a \text{ are positive constants.}
\]

(a) Let \( f(y) = (r - ay)y \). Sketch \( f(y) \) versus \( y \) for the case \( r = 2 \) and \( a = 1 \). Use the sketch to determine all equilibrium points of the differential equation and to label them as stable or unstable. (Give brief reasons for your choices.)

\[
f(y) = (r - ay)y
\]

**Zeros:**

\[
(r - ay)y = 0
\]

\[
\Rightarrow y = 0, \quad y = \frac{r}{a}
\]

**Stability:**

\[
f'(y) = r - ay + y(-a)
\]

\[
= r - 2ay
\]

\[
f'(0) = r - 2a > 0 \Rightarrow \text{unstable}
\]

\[
f'(\frac{r}{a}) = r - 2a\left(\frac{r}{a}\right) = -r < 0 \Rightarrow \text{stable}
\]
2. (b) Recall again the population model

\[ \frac{dy}{dt} = (r - ay)y, \quad \text{where } r \text{ and } a \text{ are positive constants.} \]

It is known that (i) the population increases at a rate of 50 insects per hour when the population is equal to 100 insects and (ii) the population settles to 600 insects after a very long time. Find \( r \) and \( a \) for this case.

\[ i) \quad y' = 50 \quad \text{when } y = 100 \]

\[ ii) \quad y \to 600 \quad \text{as } t \to \infty \]

or, \( y' = 0 \) \quad \text{when } y = 600

So,

\[ i) \quad 50 = (r - 100a) \cdot 100 \]

\[ r = \frac{1}{2} + 100a \]

\[ ii) \quad 0 = (r - 600a) \cdot 600 \]

\[ \frac{1}{2} + 100a - 600a = 0 \]

\[ -500a = -\frac{1}{2} \]

\[ a = +\frac{1}{1000} \]

\[ r = \frac{1}{2} + \frac{1}{10} = \frac{3}{5} \]
3. Find general solutions for the following second-order equations:

(a) \(25y'' - 30y' + 9y = 0\),  
(b) \(2y'' + 2y' + 3y = 0\)

\[a) \quad 25r^2 - 30r + 9 = 0 \]

\[r = \frac{30 \pm \sqrt{900 - 4(9)(25)}}{50} = \frac{3}{5} \pm 0 \]

\[= \frac{3}{5}, \frac{3}{5} \]

\[y(t) = c_1 e^{\frac{3}{5}t} + c_2 t e^{\frac{3}{5}t} \]

\[b) \quad 2r^2 + 2r + 3 = 0 \]

\[r = -1 \pm \frac{\sqrt{4 - 4(3)(2)}}{4} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} i \]

\[y(t) = c_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{5}}{2}t\right) + c_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{5}}{2}t\right) \]
4. Find an interval of $t$ such that a unique solution of the initial-value problem is guaranteed to exist

$$(t^2 - 3)y'' + \frac{ty'}{t+1} + e^t y = 0, \quad y(0) = 0, \quad y'(0) = -2$$

You need not find the solution, but you should give reasons for your choice of an interval of $t$.

**Theorem**

Let $p(t)$, $q(t)$, and $g(t)$ be continuous on $[a, b]$, then the differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

$y(a) = y_0$

$y'(a) = y'_0$

has a unique solution defined for all $t$ in $[a, b]$. 

$$y'' + \frac{ty'}{(t+1)(t^2-3)} + \frac{e^t}{t^2-3} y = 0$$

$p$, $q$, $g$ continuous except at $t = -1$.

$t = \pm \sqrt{3} \implies [0, \sqrt{3})$
5. A student takes out a loan for $5000 just before coming to RPI to help her parents pay the cost of tuition. The annual interest rate on the loan is fixed at 5% which is compounded continuously.

(a) Determine how much the student owes after 4 years at RPI.

(b) Assume the student owes N dollars after 4 years at RPI. The student then gets a good job and begins paying off the loan at a continuous rate of $600 per year. (The interest rate is still 5%.) Determine how long it will take for the student to pay off the loan after leaving RPI.

\[ \frac{dQ}{dt} = 0.05Q, \quad Q(0) = 5000 \]

\[ \int \frac{dQ}{Q} = \int 0.05 dt \]

\[ \ln Q = 0.05t + C \]

\[ Q = Ce^{0.05t} \]

\[ Q(0) = C = 5000 \quad \Rightarrow \quad Q(t) = 5000e^{0.05t} \]

b) \[ \frac{dQ}{dt} = 0.05Q - 600 \]

\[ Q(4) = N = 5000e^{-2} \]

\[ Q' = 0.05Q = -600 \]

\[ \frac{d}{dt}(Qe^{-0.05t}) = -600e^{-0.05t} \]
\[ Q e^{-0.05t} = -600 \cdot (20) e^{-0.05t} + C \]

\[ Q = 12000 + C e^{-0.05t} \]

\[ Q(4) = 12000 + C e^{-0.2} = 5000 e^{-2} \]

\[ C = 5000 - 12000 e^{-2} \]

\[ Q(t) = 12000 + (5000 - 12000 e^{-2}) e^{-0.05t} \]

\[ Q(t) = 0 = 12000 + (5000 - 12000 e^{-2}) e^{-0.05t} \]

\[ -\frac{12000}{5000 - 12000 e^{-2}} = e^{-0.05t} \]

\[ \ln \left( \frac{-12000}{5000 - 12000 e^{-2}} \right) = 0.05t \]

\[ t = 20 \ln \left( \frac{-12000}{5000 - 12000 e^{-2}} \right) \times 4.341 \text{ years} \]