Problem Set 9 Solutions

Due: November 23, 2015

1 Problem 1

First we find the eigenvalues and eigenvectors of the matrix

$$\det\left( \begin{bmatrix} 7 - \lambda & 9 \\ -6 & -8 - \lambda \end{bmatrix} \right) = 0$$

(1)

$$\lambda_1 = -2$$

(2)

$$\lambda_2 = 1$$

(3)

$$v_1 = [-1, 1]^\top$$

(4)

$$v_2 = [-3, 2]^\top$$

(5)

We now use the eigenvalues and vectors to form the general solution

$$x = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

(6)

$$x = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^t$$

(7)

Now we use the initial condition to solve for $c_1, c_2$

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

(8)

It is easy to see we get a solution with $c_1 = c_2 = -1$

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^t$$

(9)

As $t \to \infty$ the solution tends to infinity in the direction of $-v_2 = [3, -2]^\top$ in the $x_1 - x_2$ plane
2 Problem 2

Again we find eigenvalues and eigenvectors of the matrix. Skipping the details here we get

$$\lambda_1 = -1 + i$$  \hspace{1cm} (10)
$$\lambda_2 = -1 - i$$  \hspace{1cm} (11)
$$v_1 = [2 + i, 5]$$  \hspace{1cm} (12)
$$v_2 = [2 - i, 5]$$  \hspace{1cm} (13)

Since we desire real solutions so we will consider only one of the solutions as in example 1 of section 7.6

$$x^{(1)}(t) = \begin{bmatrix} 2 + i \\ 5 \end{bmatrix} e^{(-1+i)t}$$  \hspace{1cm} (14)

$$x^{(1)} = \begin{bmatrix} 2 + i \\ 5 \end{bmatrix} e^{-t}(\cos(t) + i \sin(t))$$  \hspace{1cm} (15)

$$x^{(1)} = \begin{bmatrix} 2 \cos(t) - \sin(t) \\ 5 \cos(t) \end{bmatrix} e^{-t} + i \begin{bmatrix} 2 \sin(t) + \cos(t) \\ 5 \sin(t) \end{bmatrix} e^{-t}$$  \hspace{1cm} (16)

Now we take the real and imaginary parts to develop the general solution.

$$x = c_1 \begin{bmatrix} 2 \cos(t) - \sin(t) \\ 5 \cos(t) \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \sin(t) + \cos(t) \\ 5 \sin(t) \end{bmatrix} e^{-t}$$  \hspace{1cm} (17)

3 Problem 3

3.1 Part a

$$x'_1 = x_2$$  \hspace{1cm} (18)
$$x'_2 = -\gamma/m x_2 - k/m x_1$$  \hspace{1cm} (19)

$$x' = \begin{bmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{bmatrix} x = Ax$$  \hspace{1cm} (20)

3.2 Part b

3.2.1 i

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$  \hspace{1cm} (21)
$$\lambda_1 = -3, v_1 = [-1, 3]^T$$  \hspace{1cm} (22)
$$\lambda_2 = -1, v_2 = [-1, 1]^T$$  \hspace{1cm} (23)
\[ x(t) = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \] (24)

### 3.2.2 ii

\[ A = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \] (25)

\[ \lambda_1 = -2 + i, \quad v_1 = [-2 - ii, 5]^T \] (26)

\[ \lambda_2 = -2 - i, \quad v_2 = [-2 + i, 5]^T \] (27)

\[ x^{(1)}(t) = \begin{bmatrix} -2 - i \\ 5 \end{bmatrix} e^{-2t} (\cos(t) + i \sin(t)) \] (28)

\[ = \begin{bmatrix} -2 \cos(t) + \sin(t) \\ 5 \cos(t) \end{bmatrix} e^{-2t} + i \begin{bmatrix} -2 \sin(t) - \cos(t) \\ 5 \sin(t) \end{bmatrix} e^{-2t} \] (29)

\[ x(t) = c_1 \begin{bmatrix} -2 \cos(t) + \sin(t) \\ 5 \cos(t) \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -2 \sin(t) - \cos(t) \\ 5 \sin(t) \end{bmatrix} e^{-2t} \] (30)

### 3.3 Part c

Solutions have some spiral like behavior in part ii.